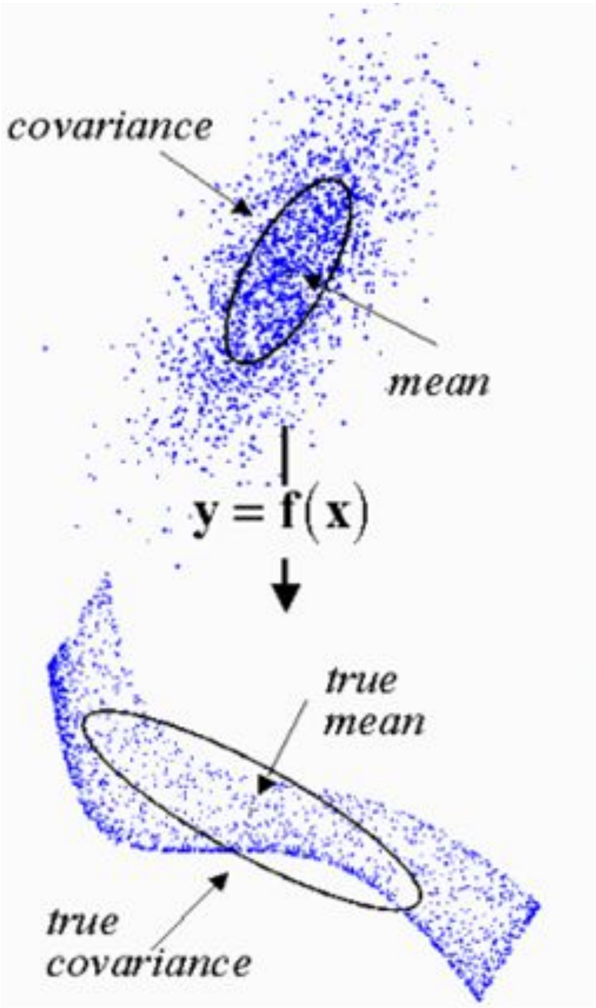
2.160 Identification, Estimation, and Learning

Part 2 Estimation

Lecture 9

Extended Kalman Filter and Unscented Kalman Filter



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7.1 Applying Kalman Filter to Nonlinear Dynamical Systems

- ❑ So far, we have been dealing with linear dynamical systems for constructing Kalman Filter.
- ❑ However, practical systems are nonlinear to some extent.

Example

Vehicle kinematics is nonlinear

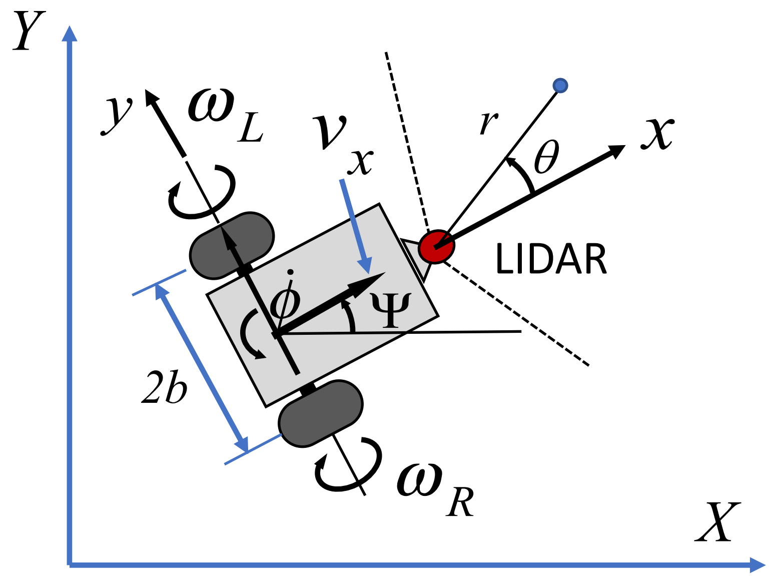
$$v_x = \frac{D}{4}(\omega_R + \omega_L) \qquad \dot{X} = v_x \cos\Psi$$

$$v_y = 0 \qquad \dot{Y} = v_x \sin\Psi$$

$$\dot{\phi} = \frac{D}{4b}(\omega_R - \omega_L) \qquad \dot{\Psi} = \dot{\phi}$$

Nonlinear State Equation

$$\longrightarrow \quad \dot{x} = f(x,u,t) + w(t)$$



LIDAR (a range finder) attached to a vehicle is in a polar coordinate system: nonlinear.

$$x = r\cos\theta$$

$$y = r\sin\theta$$

Nonlinear Measurement Equation

$$\longrightarrow \quad y(t) = h(x,t) + v(t)$$

Extension of Kalman Filter to Nonlinear Dynamical Systems

State Equation $\dot{x} = f(x,u,t) + w(t), \quad x \in \Re^{n\times 1}, u \in \Re^{r\times 1}, w \in \Re^{n\times 1}$

Measurement Equation $y = h(x,t) + v(t), \quad y \in \Re^{\ell\times 1}, v \in \Re^{\ell\times 1}$

Process noise and measurement noise are uncorrelated, white noise, with variance, $Q(\mathrm{t})$ and $R(\mathrm{t})$.

We consider three methods

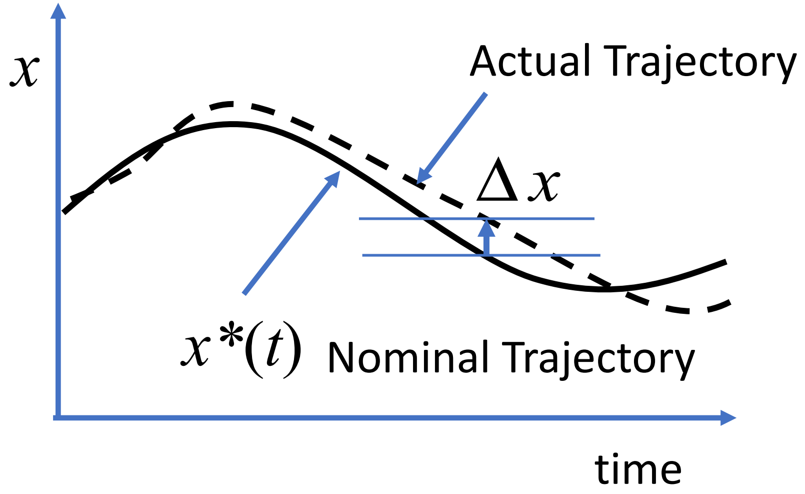
- ❑Linearized Kalman Filter
- ❑Extended Kalman Filter
- ❑Unscented Kalman Filter

7.2 Linearized Kalman Filter

- Linearize the nonlinear state equation around a nominal trajectory, e.g. reference trajectory, planned trajectory, commanded trajectory.

- The nominal trajectory must satisfy the original nonlinear state equation.

$$\dot{x}^* = f(x^*, u, t)$$



- Consider deviation from the nominal trajectory:

$$x = x^* + \Delta x$$

- Assuming that the deviation from the nominal trajectory is kept small, we can linearize the nonlinear state equation:

$$\dot{x} = f(x, u, t) + w(t) = f(x^* + \Delta x, u, t) + w(t)$$

$$= f(x^*, u, t) + \left.\frac{\partial f}{\partial x}\right|_{x^*} \Delta x + w(t) + (\textit{higher-order small quantity})$$

$$\dot{x}^* = f(x^*, u, t)$$

Linearized Kalman Filter

- Note that the nominal trajectory satisfies the state equation with no process noise and that the derivative of the deviation is given by .

$$\dot{x} = f(x^*, u, t) + \left.\frac{\partial f}{\partial x}\right|_{x^*} \Delta x + w(t)$$

$$\dot{x} = \dot{x}^* + \Delta \dot{x} \qquad \dot{x}^* = f(x^*, u, t)$$

$$\Delta \dot{x} \cong \left.\frac{\partial f}{\partial x}\right|_{x^*} \Delta x + w(t)$$

- Now replacing $\Delta \dot{x}$ by $\dot{x}$ and the Jacobian matrix $\left.\frac{\partial f}{\partial x}\right|_{x^*}$ by F(t), we have a linear time-varying state equation

$$\dot{x} = F(t)x(t) + w(t)$$

Linearized Kalman Filter

- Similarly, the measurement equation can be linearized around the nominal trajectory:

$$y^* + \Delta y \cong h(x^*, t) + \left.\frac{\partial h}{\partial x}\right|_{x^*} \Delta x + v(t)$$

- Note again $y^* = h(x^*, t)$. Replacing Δy by y and the Jacobian matrix $\left.\frac{\partial h}{\partial x}\right|_{x^*}$ by H(t),

$$y(t) = H(t)x(t) + v(t)$$

- The original Linear Kalman Filter can be applied to this linear <u>time-varying</u> system.

State Equation $\dot{x} = f(x, u, t) + w(t), \Rightarrow \quad \dot{x} = F(t)x(t) + w(t) \qquad F(t) = \left.\frac{\partial f}{\partial x}\right|_{x^*}$

Measurement Equation $y = h(x, t) + v(t), \quad \Rightarrow \quad y(t) = H(t)x(t) + v(t) \qquad H(t) = \left.\frac{\partial h}{\partial x}\right|_{x^*}$

State propagation and update $\frac{d}{dt}\hat{x}(t) = F(t)\hat{x}(t) + K(t)[y(t) - \hat{y}(t)] \qquad K(t) = P(t)H^T(t)R^{-1}(t)$

Riccati Differential Equation $\frac{dP(t)}{dt} = F(t)P(t) + P(t)F^T(t) - P(t)H^T(t)R^{-1}(t)H(t)P(t) + G(t)Q(t)G^T(t)$

(Covariance propagation and update)

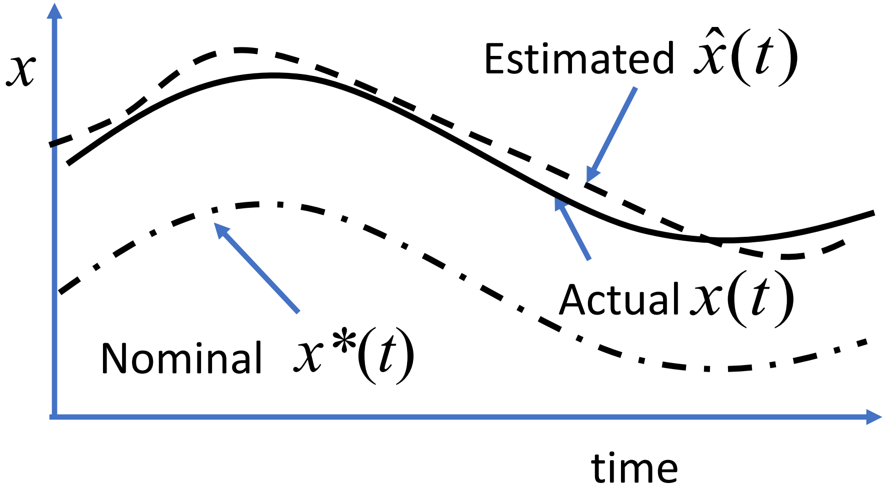
7.3 Extended Kalman Filter

- Linearized Kalman Filter is simple, but performs poorly when the actual trajectory deviates from a nominal trajectory. The state transition and measurement are linear approximation, while the true system is nonlinear.
- Extended Kalman Filter is a significant improvement in two major aspects:

1) The Jacobian matrices are evaluated not at nominal state $x^*(t)$ but at an estimated state $\hat{x}(t)$

$$F(t) = \left.\frac{\partial f}{\partial x}\right|_{\hat{x}(t)} \qquad \cancel{F(t) = \left.\frac{\partial f}{\partial x}\right|_{x^*}}$$

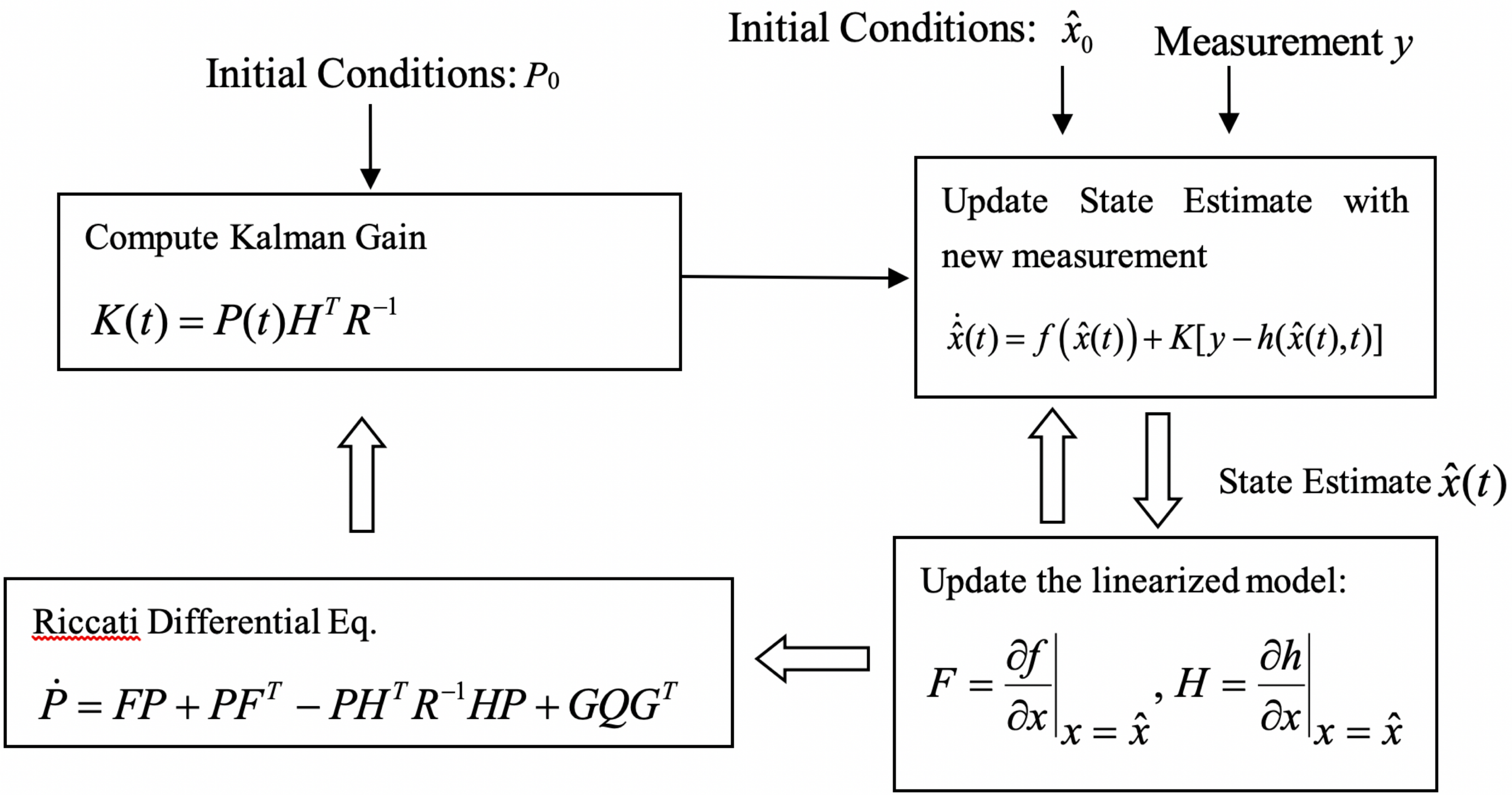
$$H(t) = \left.\frac{\partial h}{\partial x}\right|_{\hat{x}(t)} \qquad \cancel{H(t) = \left.\frac{\partial h}{\partial x}\right|_{x^*}}$$



2) State propagation and update use the full nonlinear state equation and measurement equation.

$$\dot{\hat{x}} = f(\hat{x}(t),t) + K(t)[y(t) - \underbrace{h(\hat{x}(t),t)}_{\hat{y}(t)}] \qquad \cancel{\dot{\hat{x}} = F(t)\hat{x}(t) + K(t)[y(t) - H(t)\hat{x}(t)]}$$

Extended Kalman Filter



Extended Kalman Filter

- Covariance propagation and update, however, are based on the linearized model of the nonlinear system. Namely, we use the Riccati Differential Equation using the Jacobian matrices evaluated at estimated state $\hat{x}(t)$.

$$\frac{dP(t)}{dt} = \underline{F(t)}P(t) + P(t)\underline{F^T(t)} - P(t)\underline{H^T(t)}R^{-1}(t)\underline{H(t)}P(t) + G(t)Q(t)G^T(t)$$

First-order approximation of nonlinear functions

- Extended Kalman Filter is a nonlinear filter.
- Optimality of state estimation is no longer guaranteed.
- Extended Kalman Filter (EKF) tends to underestimate the error covariance, when the system is highly nonlinear. This sometimes causes the divergence of estimate.

Divergence Scenario

Underestimated P → Small Kalman Gain K → Insufficient State Update → Growing estimation error → Inaccurate Jacobians F(t) and H(t) → Blow up

- Unscented Kalman Filter can better handle such a problem.

7.4 Unscented Transform

- Unscented Kalman Filter, originally developed by Julier and Uhlmann [1997], uses a different method for computing error covariance matrices.

- It does not use the Riccati Equations (continuous time) or the covariance propagation and update laws (discrete time). Instead it uses a special technique, called Unscented Transform, for propagating and updating covariance matrices.

- The key idea is to estimate the error covariance based on a special set of sample points, termed “sigma points”, which propagate directly through the original nonlinear model.

7.4 Unscented Transform

- Consider a simple case where one-dimensional random variable X has a Gaussian distribution.

- Mean $\bar{x}$ and variance σ^2 completely characterize the distribution.

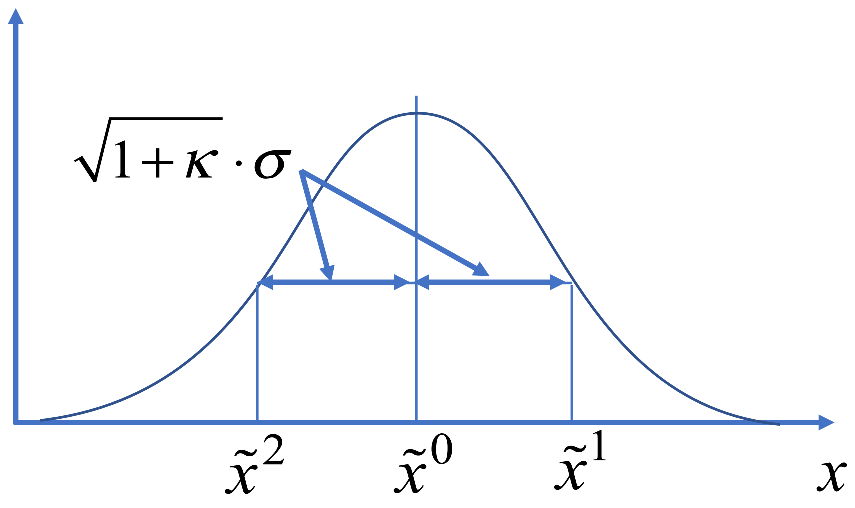
$$p(x) = \frac{1}{\sqrt{2\pi}\cdot\sigma}\exp\left(-\frac{(x-\bar{x})^2}{\sigma^2}\right)$$

- Unscented transform represents Gaussian distribution with three special sample points, called Sigma Points.

$$\tilde{x}^0 = \bar{x} \qquad W_0 = \frac{\kappa}{1+\kappa}$$

$$\tilde{x}^1 = \bar{x} + \sqrt{1+\kappa}\cdot\sigma$$

$$\tilde{x}^2 = \bar{x} - \sqrt{1+\kappa}\cdot\sigma \qquad W_1 = W_2 = \frac{1}{2(1+\kappa)}$$



Sigma Points

where κ is a parameter of sigma points to be tuned, and W_i is the weight of the i^{th} sigma point used for computing mean and variance.

Unscented Transform

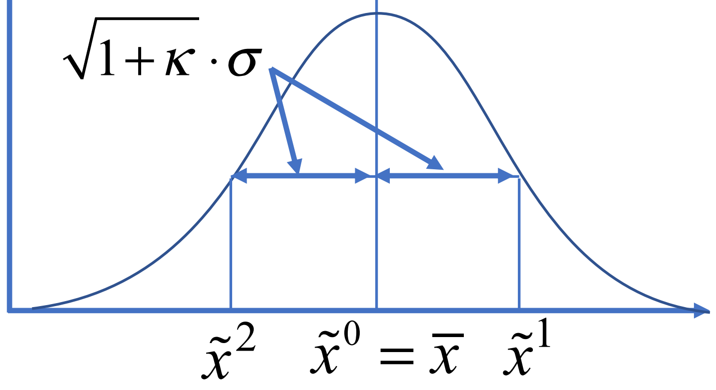
$$W_0 = \frac{\kappa}{1+\kappa}$$

$$W_1 = W_2 = \frac{1}{2(1+\kappa)}$$

- The weighted mean of the three Sigma points agrees with the true mean of the Gaussian:

$$\sum_{i=0}^{2} W_i \tilde{x}^i = \frac{\kappa}{1+\kappa}\bar{x} + \frac{1}{2(1+\kappa)}\left\{(\bar{x}+\sqrt{1+\kappa}\cdot\sigma)+(\bar{x}-\sqrt{1+\kappa}\cdot\sigma)\right\}$$

$$= \frac{\kappa}{1+\kappa}\bar{x} + \frac{2}{2(1+\kappa)}\bar{x} = \bar{x}$$



- The weighted variance of the three Sigma points agrees with the true variance:

$$\sum_{i=0}^{2} W_i (\tilde{x}^i - \bar{x})^2 = \frac{\kappa}{1+\kappa}(\bar{x}-\bar{x}) + \frac{1}{2(1+\kappa)}\left\{(\bar{x}+\sqrt{1+\kappa}\cdot\sigma-\bar{x})^2+(\bar{x}-\sqrt{1+\kappa}\cdot\sigma-\bar{x})^2\right\}$$

$$= \frac{2}{2(1+\kappa)}(\sqrt{1+\kappa}\cdot\sigma)^2 = \sigma^2$$

- Therefore, the mean and variance of Sigma points provide the correct mean and variance of the true Gaussian distribution for an arbitrary value of parameter κ.

Unscented Transform

- Now consider a nonlinear transformation from x to y,

$$y = g(x) = \sum_{k=0}^{\infty} \frac{g^{(k)}(\bar{x})}{k!}(x-\bar{x})^k$$

where function $g(x)$ is analytic.

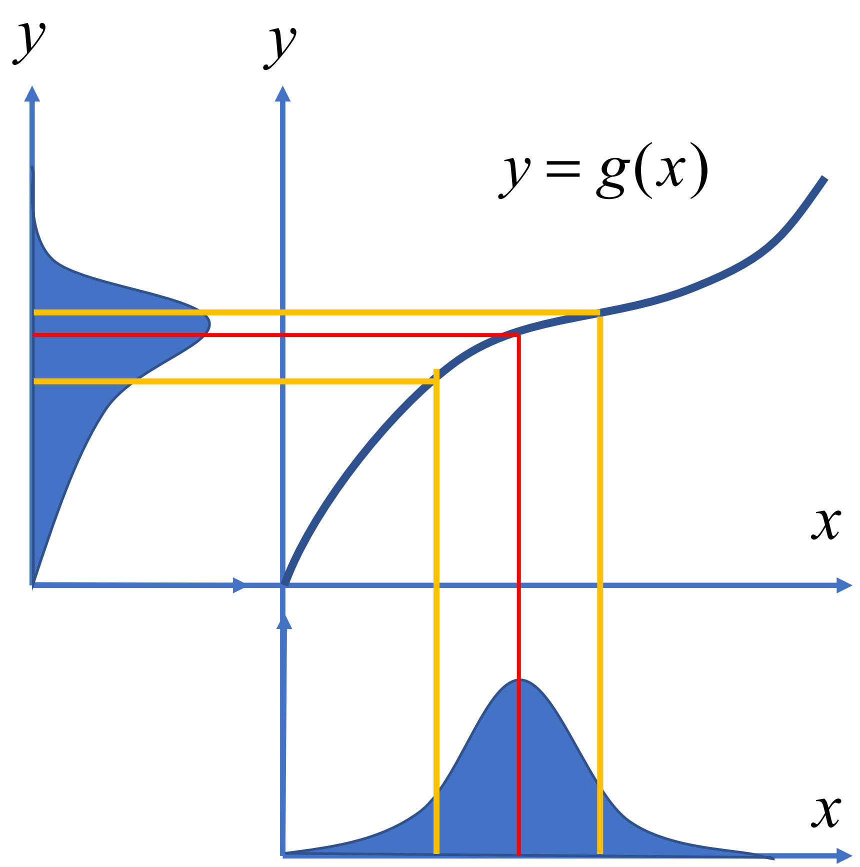
- The distribution of y is no longer Gaussian, but its mean $E[y]$ and variance $E[(y\text{-}E[y])^2]$ can characterize the distribution.
- We can show that the weighted mean of Sigma points transformed through the nonlinear function can approximate the true mean to the third order.

$$\bar{y}_{sample} \triangleq \sum_{i=0}^{2} W_i \tilde{y}^i = E[y] + O(4)$$

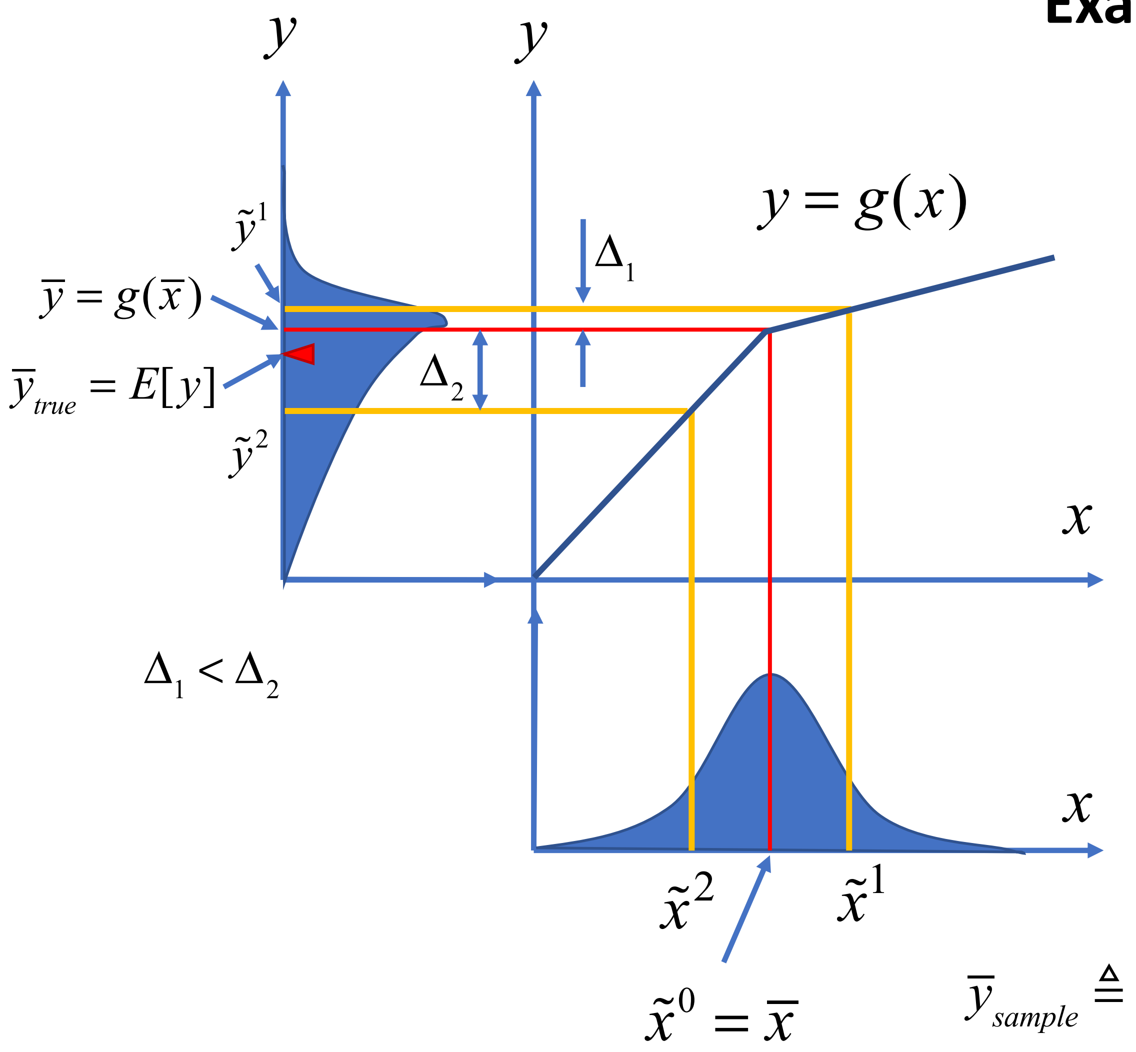
where $O(4)$ is a small quantity of order 4 and higher.

- Furthermore, we can show that the weighted variance can approximate the true variance to the second order.

$$\sigma^2_{sample} \triangleq \sum_{i=0}^{2} W_i (\tilde{y}^i - \bar{y}_{sample})^2 = E[(y - E[y])^2] + \mathrm{O}(3)$$



Example



- Consider a piecewise linear function for $y = g(x)$, as shown in the figure. (The two lines are connected with a smooth curve for differentiability.)
- The mean of x is transformed to $\bar{y} = g(\bar{x})$, which is not the true mean of the distribution of y. We can see by inspection:

$$\bar{y}_{true} = E[y] < \bar{y} = g(\bar{x})$$

- The weighted mean of the Sigma points can give a better mean value.

$$\bar{y}_{sample} \triangleq \sum_{i=0}^{2} W_i \tilde{y}^i = \frac{\kappa}{1+\kappa}\tilde{y}^0 + \frac{1}{2(1+\kappa)}(\tilde{y}^1 + \tilde{y}^2)$$

$$= \frac{\kappa}{1+\kappa}\bar{y} + \frac{1}{2(1+\kappa)}(\bar{y} + \Delta_1 + \bar{y} - \Delta_2) = \bar{y} + \frac{1}{2(1+\kappa)}(\Delta_1 - \Delta_2) < \bar{y}$$

The weighted mean of the Sigma points gives a better mean value. $\because \Delta_1 < \Delta_2$

Sigma Points for Multivariate Gaussian Distribution

- In general, for an n-dimensional Gaussian distribution, we use $(2n+1)$ Sigma points.

$$p(x) = \det(2\pi \mathbf{P}_x)^{-\frac{1}{2}} \exp\left\{ -\frac{1}{2}(x-\bar{x})^T \mathbf{P}_x^{-1}(x-\bar{x}) \right\}$$

- Note that the covariance matrix P_x is real, symmetric, and positive-definite. Therefore, it can be diagonalized

$$\mathbf{P}_x = \mathbf{V}\mathbf{D}\mathbf{V}^T$$

where $\mathbf{V} = (\mathbf{v}_1 \cdots \mathbf{v}_n),\ \mathbf{D} = diag(\sigma_1^2 \cdots \sigma_n^2)$

- Sigma points are taken along the individual eigen vectors with unit length, $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_\text{n}$.

$$\tilde{x}^0 = \bar{x}$$

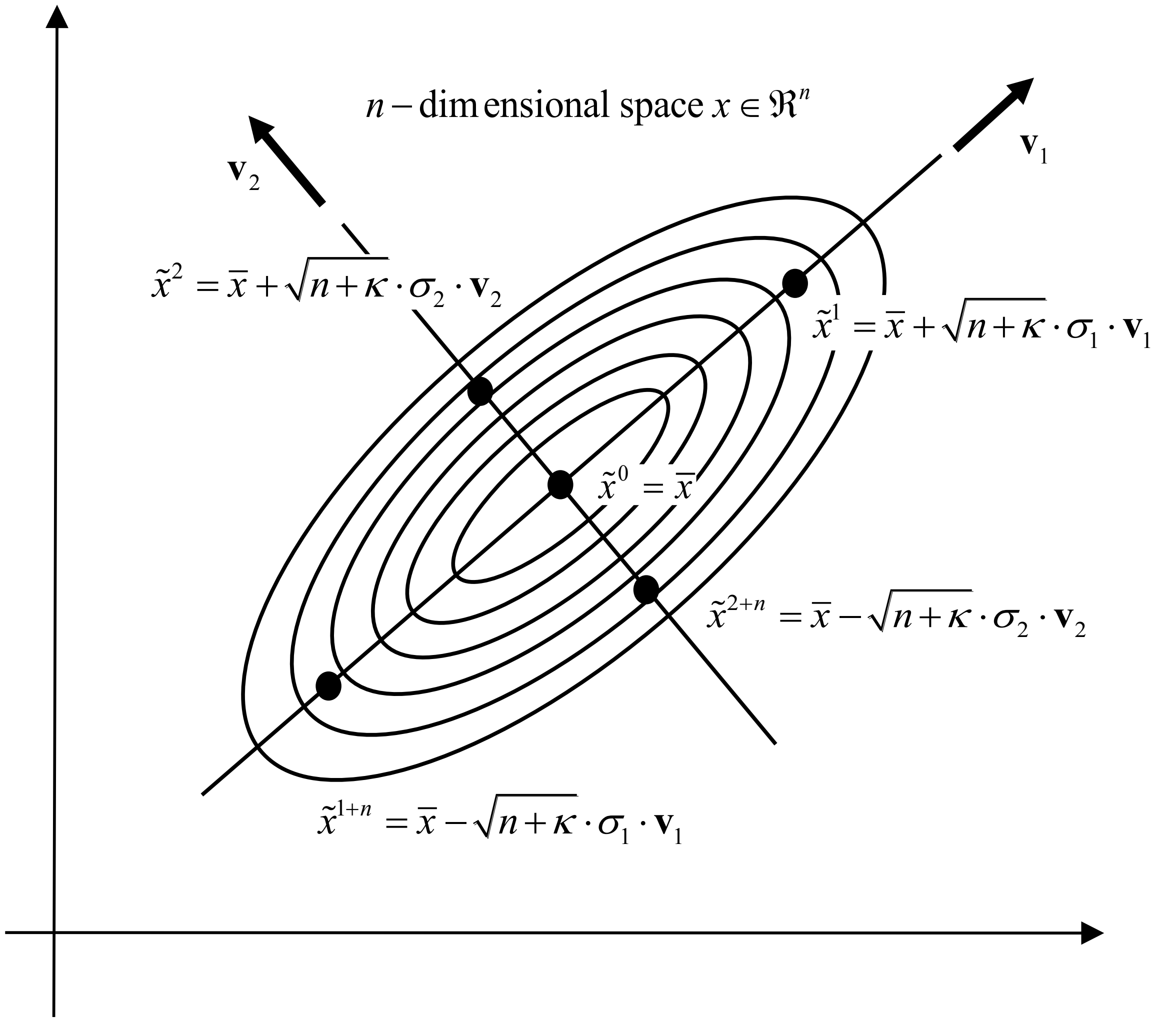
$$\tilde{x}^i = \bar{x} + \sqrt{n+\kappa} \cdot \sigma_i \cdot \mathbf{v}_i$$

$$\tilde{x}^{i+n} = \bar{x} - \sqrt{n+\kappa} \cdot \sigma_i \cdot \mathbf{v}_i$$

$$W_0 = \frac{\kappa}{n+\kappa}$$

$$W_i = W_{i+n} = \frac{1}{2(n+\kappa)}$$

$$i = 1, \cdots, n$$



Sigma Points for Multivariate Gaussian Distribution

- As before, all the Sigma points propagate through a nonlinear analytic function, $y = g(x)$, which is multivariate.

$$\tilde{y}^i = g(\tilde{x}^i), \quad i = 0, \cdots, 2n$$

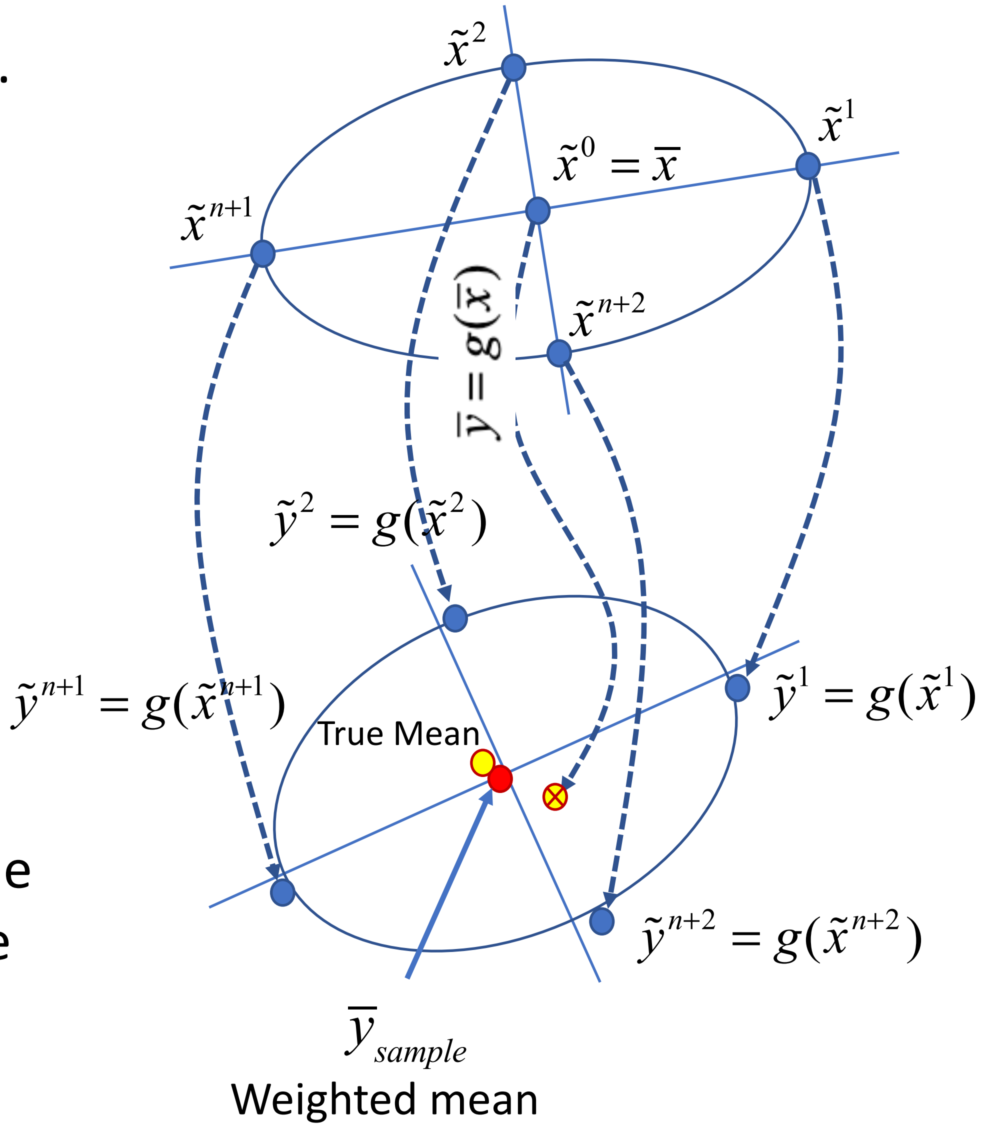
- For these propagated points, the weighted mean is computed:

$$\bar{y}_{sample} = \sum_{i=0}^{2n} W_i \, \tilde{y}^i$$

- The weighted covariance is given by

$$P_{y,sample} = \sum_{i=0}^{2n} W_i (\tilde{y}^i - \bar{y}_{sample})(\tilde{y}^i - \bar{y}_{sample})^T$$

- We can show that the weighted mean can approximate the true mean to the third order, and the weighted covariance to the second order.
- This sampling method is called ***Unscented Transform***.



7.5 Unscented Kalman Filter

- Consider a nonlinear discrete-time dynamical system:

$u_t = 0$

$$x_{t+1} = f(x_t, \cancel{u_t}, t) + w_t$$ without loss of generality we set u to zero.

$$y_t = h(x_t, t) + v_t$$

- Process noise and measurement noise are zero-mean, uncorrelated (white) noise with covariance Q_t and R_t.

$$w_t \sim N(0, Q_t), \quad v_t \sim N(0, R_t)$$

- At time t-1, a posteriori estimate $\hat{x}_{t-1}$ and a posterior error covariance are available:

$$P_{t-1} = E[(\hat{x}_{t-1} - x_{t-1})(\hat{x}_{t-1} - x_{t-1})^T]$$

- The problem is to find a recursive formula using Unscented Transform:
  - Propagation of state and covariance: Find $\hat{x}_{t|t-1}$ and $P_{t|t-1}$ from $\hat{x}_{t-1}$ and. P_{t-1};
  - Update of state and covariance: Find $\hat{x}_t$ and P_t from $\hat{x}_{t|t-1}$ and $P_{t|t-1}$.

Propagation of State

- Find eigenvalues and eigen-vectors of covariance P_{t-1} and generate $(2n + 1)$ Sigma points

$$\tilde{x}^0_{t-1} = \hat{x}_{t-1} : \quad \bar{x}$$

$$\tilde{x}^i_{t-1} = \hat{x}_{t-1} + \sqrt{n+\kappa}\cdot\sigma_i\cdot\mathbf{v}_i, \quad \tilde{x}^{i+n}_{t-1} = \hat{x}_{t-1} - \sqrt{n+\kappa}\cdot\sigma_i\cdot\mathbf{v}_i$$

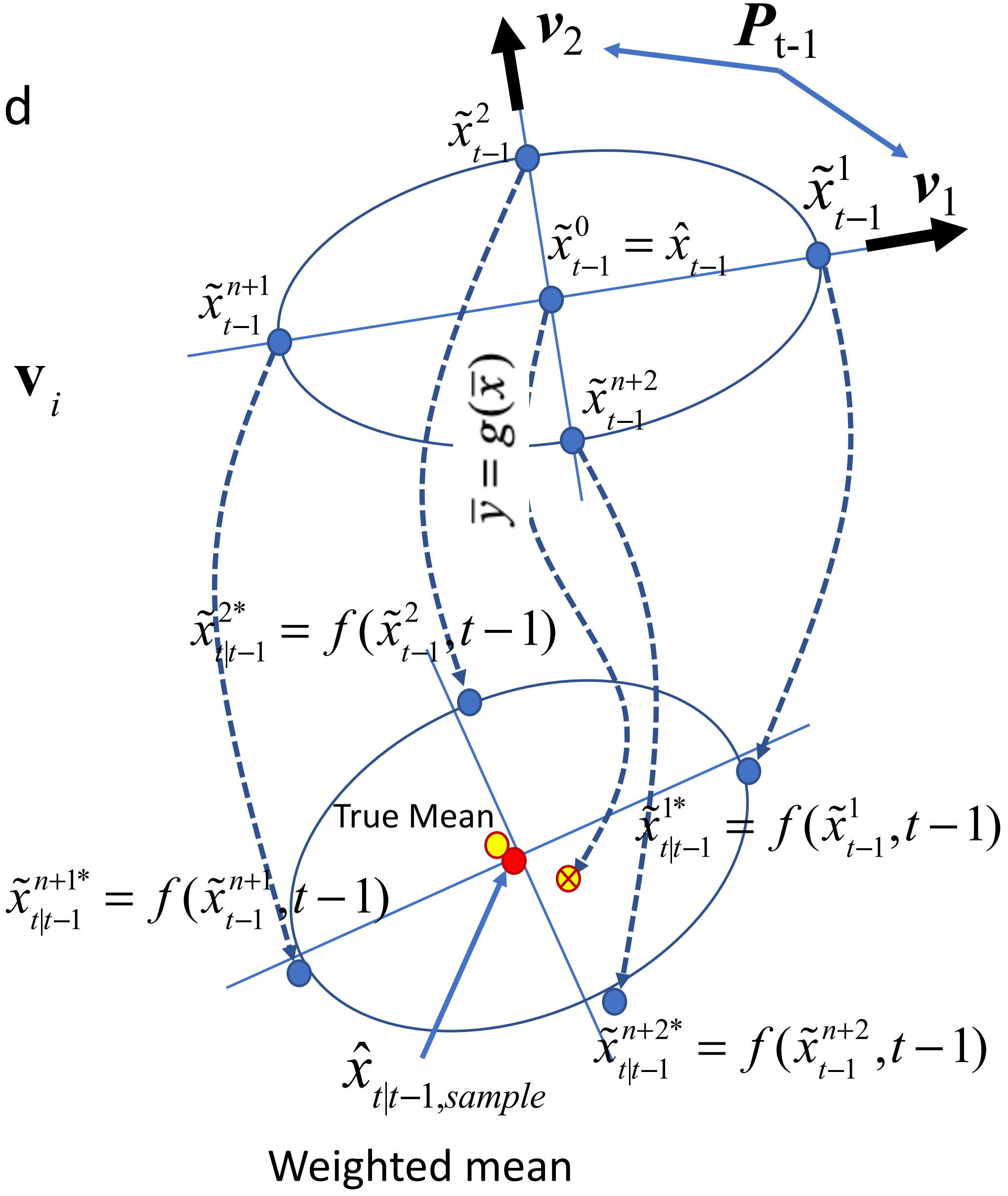
$$i = 1,\cdots,n$$

- Propagate the Sigma points through the state equation, noting that the process noise is zero mean.

$$\tilde{x}^{i*}_{t|t-1} = f(\tilde{x}^i_{t-1}, t-1) + \cancelto{0}{w_{t-1}}, \quad i = 0,\cdots,2n$$

- For these (2n+1) Sigma points, the weighted mean is computed:

$$\hat{x}_{t|t-1,sample} = \sum_{i=0}^{2n} W_i\,\hat{x}^{i*}_{t|t-1}$$



Propagation of Covariance

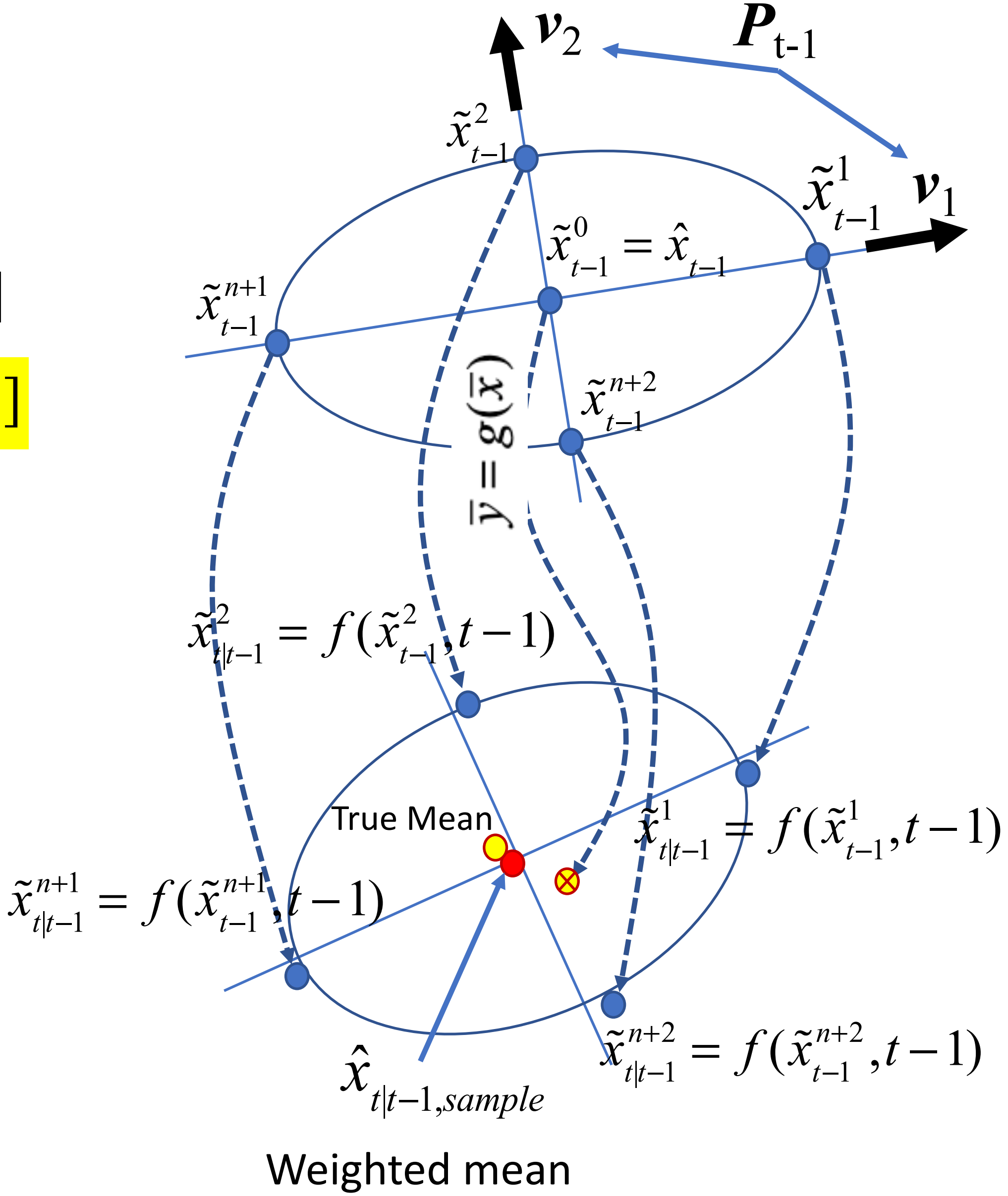
- a priori covariance is computed as

$$
\begin{aligned}
P_{t|t-1} &= E[(\hat{x}_{t|t-1} - x_t)(\hat{x}_{t|t-1} - x_t)^T] \\
&= E[(\hat{x}_{t|t-1} - f(x_{t-1}, t-1) - w_{t-1})(\hat{x}_{t|t-1} - f(x_{t-1}, t-1) - w_{t-1})^T] \\
&= E[(\hat{x}_{t|t-1} - f(x_{t-1}, t-1))(\hat{x}_{t|t-1} - f(x_{t-1}, t-1))^T] + \underbrace{E[w_{t-1}w_{t-1}^T]}_{Q_{t-1}} \\
&\quad + \cancel{(cross - terms)} \rightarrow 0
\end{aligned}
$$

- The term $f(x_{t-1}, t-1)$ is not known since we do not know the exact state x_{t-1}. However, it is approximated to the 3rd order with the weighted mean; $\hat{x}_{t|t-1,sample}$. Therefore, we replace $f(x_{t-1}, t-1)$ by the weighted mean, and compute the a priori error covariance using the (2n+1) Sigma points.

$$
P_{t|t-1,sample} = \sum_{i=0}^{i=2n} W_i (\tilde{x}^i_{t|t-1} - \hat{x}_{t|t-1,sample})(\tilde{x}^i_{t|t-1} - \hat{x}_{t|t-1,sample})^T + Q_{t-1}
$$

- Note that the predicted covariance is correct up to the second order. For brevity, the subscript "sample" will be dropped hereafter.



Obtaining the Kalman Gain from Innovation Covariance

- We aim to update state and covariance using Sigma points. The standard Kalman gain and covariance update laws, however, are not applicable to Unscented Transformation. Instead, we will consider an alternative method based on *innovation*.
- The Kalman Gain is given by $K_t = P_{t|t-1} H_t^T (H_t P_{t|t-1} H_t^T + R_t)^{-1}$, but the matrix H_t is not available for our nonlinear system.
- Instead, we use the following formula,

$$K_t = P_{xy} P_y^{-1}$$

where $P_y = E[(y_t - \hat{y}_t)(y_t - \hat{y}_t)^T], \quad P_{xy} = E[(x_t - \hat{x}_{t|t-1})(y_t - \hat{y}_t)^T]$

For a linear time-varying system, this new formula can be proven by construction.

$$\begin{aligned} P_y &\triangleq E\left[(y_t - \hat{y}_t)(y_t - \hat{y}_t)^T\right] \\ &= E\left[(H_t x_t + v_t - H_t \hat{x}_{t|t-1})(H_t x_t + v_t - H_t \hat{x}_{t|t-1})^T\right] \\ &= H_t E\left[(x_t - \hat{x}_{t|t-1})(x_t - \hat{x}_{t|t-1})^T\right] H_t^T + E\left[v_t v_t^T\right] \\ &= H_t P_{t|t-1} H_t^T + R_t \end{aligned}$$

$$\begin{aligned} P_{xy} &\triangleq E\left[(\hat{x}_{t|t-1} - x_t)(\hat{y}_t - y_t)^T\right] \\ &= E\left[(\hat{x}_{t|t-1} - x_t)(H_t \hat{x}_{t|t-1} - H_t x_t - v_t)^T\right] \\ &= E\left[(\hat{x}_{t|t-1} - x_t)(\hat{x}_{t|t-1} - x_t)^T\right] H_t^T - E\left[(\hat{x}_{t|t-1} - x_t) v_t^T\right] \\ &= P_{t|t-1} H_t^T \end{aligned}$$

Combining the above two, the new formula of the Kalman gain is proven. The covariance of output y is called "*Innovation Covariance*".

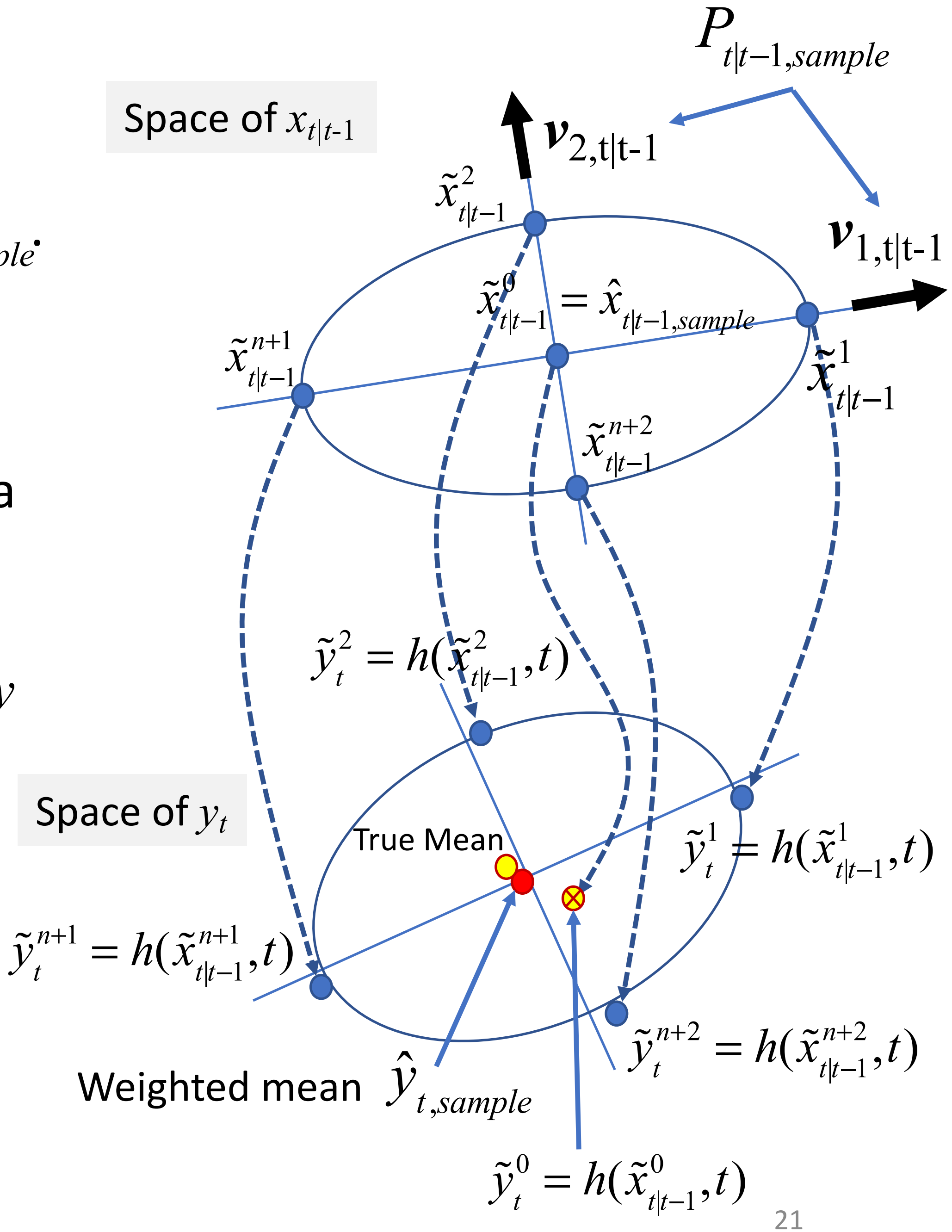
State Update

- State and covariance can be updated by applying Unscented Transform to the a priori covariance $P_{t|t-1,sample}$.
- We first obtain the innovation covariance by examining the distribution of the output created through the measurement equation.
- We compute the eigenvalues and eigen-vectors of the a priori covariance, $\sigma_{1,t|t-1}$, $\sigma_{2,t|t-1}$, ……, $\boldsymbol{v}_{1,t|t-1}$, $\boldsymbol{v}_{2,t|t-1}$, ….
- Then, we generate $(2n+1)$ Sigma points, and estimate the mean and covariance of the distribution of output y with the Sigma points.
- Using the deterministic part of the measurement function, the Sigma points are mapped to

$$\tilde{y}_t^i = h(\tilde{x}_{t|t-1}^i, t), \quad i = 0, \cdots, 2n$$

- The weighted mean of the Sigma points is given by

$$\hat{y}_{t,sample} = \sum_{i=0}^{2n} W_i \tilde{y}_t^i$$



State Update

- From the Innovation Covariance,

$$P_y \triangleq E\left[(y_t - \hat{y}_t)(y_t - \hat{y}_t)^T\right]$$
$$= E\left[(h(x_t,t) + v_t - h(\hat{x}_{t|t-1},t))(h(x_t,t) + v_t - h(\hat{x}_{t|t-1},t))^T\right]$$
$$= E\left[(h(x_t,t) - h(\hat{x}_{t|t-1},t))(h(x_t,t) - h(\hat{x}_{t|t-1},t))^T\right] + \underbrace{E\left[v_t v_t^T\right]}_{R_t}$$

- Computing the first term using Sigma points,

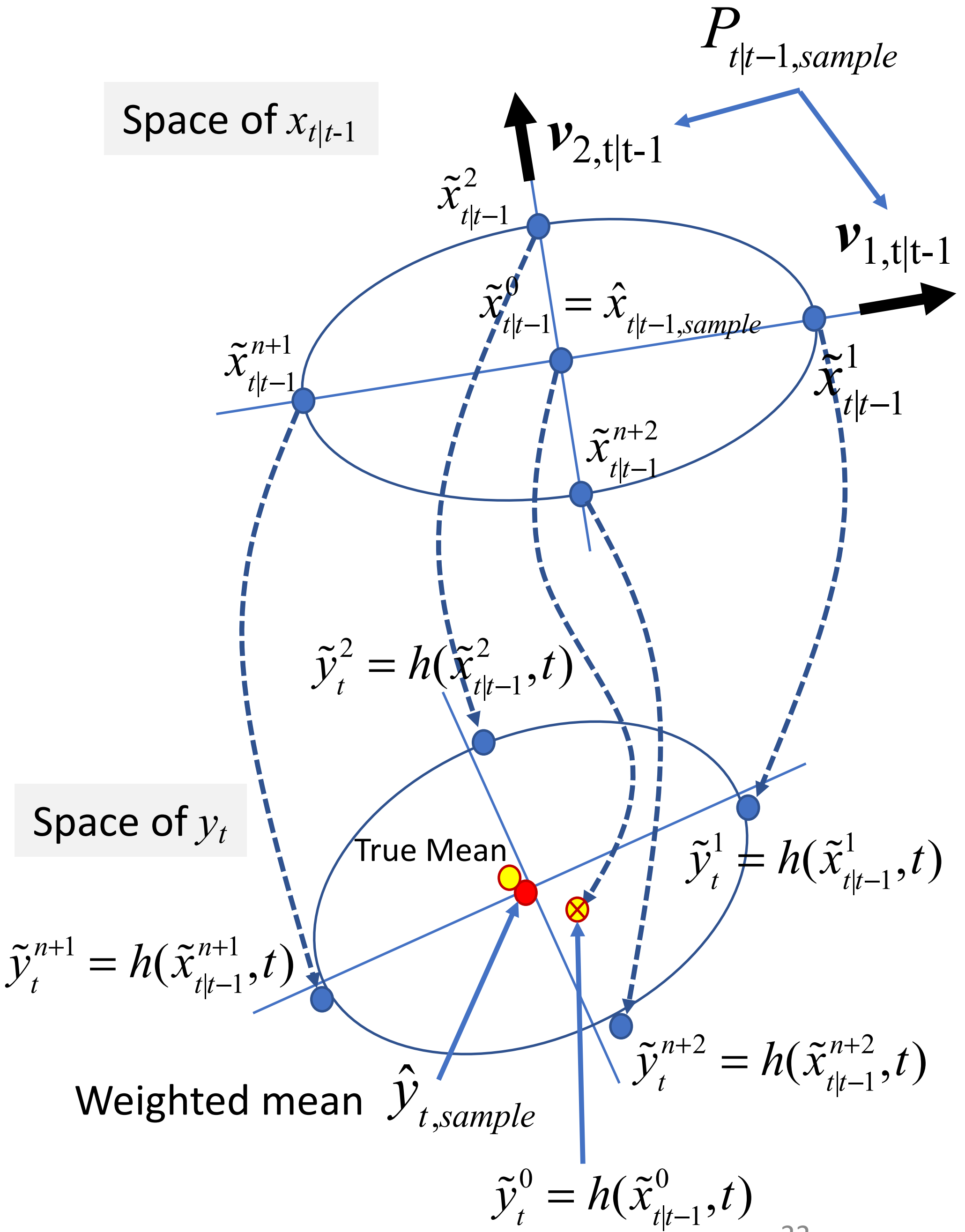
$$P_y = \sum_{i=0}^{2n} W_i (\tilde{y}_t^i - \hat{y}_{t,sample})(\tilde{y}_t^i - \hat{y}_{t,sample})^T + R_t$$

- Similarly, the cross-covariance is given by

$$P_{xy} = \sum_{i=0}^{2n} W_i (\tilde{x}_{t|t-1}^i - \hat{x}_{t|t-1,sample})(\tilde{y}_t^i - \hat{y}_{t,sample})^T$$

- The Kalman Gain is then $K_t = P_{xy} P_y^{-1}$.
- The state update is given by

$$\hat{x}_t = \hat{x}_{t|t-1,sample} + K_t [y_t - \hat{y}_{t,sample}]$$



Covariance Update

- The covariance update formula includes measurement matrix H_t, which must be replaced. We can use the innovation covariance for this:

$$P_y = E[(y_t - \hat{y}_t)(y_t - \hat{y}_t)^T]$$

- Inserting $P_y P_y^{-1} = I$ in the covariance update formula,

$$P_t = (I - K_t H_t) P_{t|t-1} = P_{t|t-1} - K_t H_t P_{t|t-1}$$

$$= P_{t|t-1} - K_t P_y P_y^{-1} H_t P_{t|t-1} = P_{t|t-1} - K_t P_y K_t^T$$

where the Kalman gain $K_t = P_{t|t-1} H_t^T P_y^{-1}$ is used to eliminate the measurement matrix H_t.

- Using the Sigma points, this can be computed as follows.

$$P_t \cong P_{t|t-1,sample} - K_t P_y K_t^T$$

The Recursive Algorithm of Unscented Kalman Filter

- Given $\hat{x}_{t-1}$ and $P_{t\text{-}1}$, sample sigma points by computing eigenvalues and eigen vectors of $P_{t\text{-}1}$;
- Propagate the sigma points through the nonlinear model to obtain $\tilde{x}^{i*}_{t|t-1} = f(\tilde{x}^{i}_{t-1}, t-1)$;
- From the (2n+1) sigma points compute the mean and variance:

$$\hat{x}_{t|t-1,sample} = \sum_{i=0}^{2n} W_i \hat{x}^{i*}_{t|t-1} \qquad P_{t|t-1,sample} = \sum_{i=0}^{i=2n} W_i (\tilde{x}^{i}_{t|t-1} - \hat{x}_{t|t-1,sample})(\tilde{x}^{i}_{t|t-1} - \hat{x}_{t|t-1,sample})^T + Q_{t-1}$$

- Sample again (2n+1) sigma points for $P_{t|t\text{-}1,sample}$;
- Transform the propagated sigma points to output estimate $\tilde{y}^{i}_{t}$ based on the nonlinear measurement equation, and compute the estimated output

$$\hat{y}_{t,sample} = \sum_{i=0}^{2n} W_i \tilde{y}^{i}_{t} \qquad \tilde{y}^{i}_{t} = h(\tilde{x}^{i}_{t|t-1}, t)$$

- Evaluate the innovation covariance and the cross covariance by using (2n+1) points of propagated output estimates to find the Kalman gain

$$K_t = P_{xy} P_y^{-1} \qquad P_y = \sum_{i=0}^{2n} W_i (\tilde{y}^{i}_{t} - \hat{y}_{t,sample})(\tilde{y}^{i}_{t} - \hat{y}_{t,sample})^T + R_t \qquad P_{xy} = \sum_{i=0}^{2n} W_i (\tilde{x}^{i}_{t|t-1} - \hat{x}_{t|t-1,sample})(\tilde{y}^{i}_{t} - \hat{y}_{t,sample})^T$$

The Recursive Algorithm of Unscented Kalman Filter

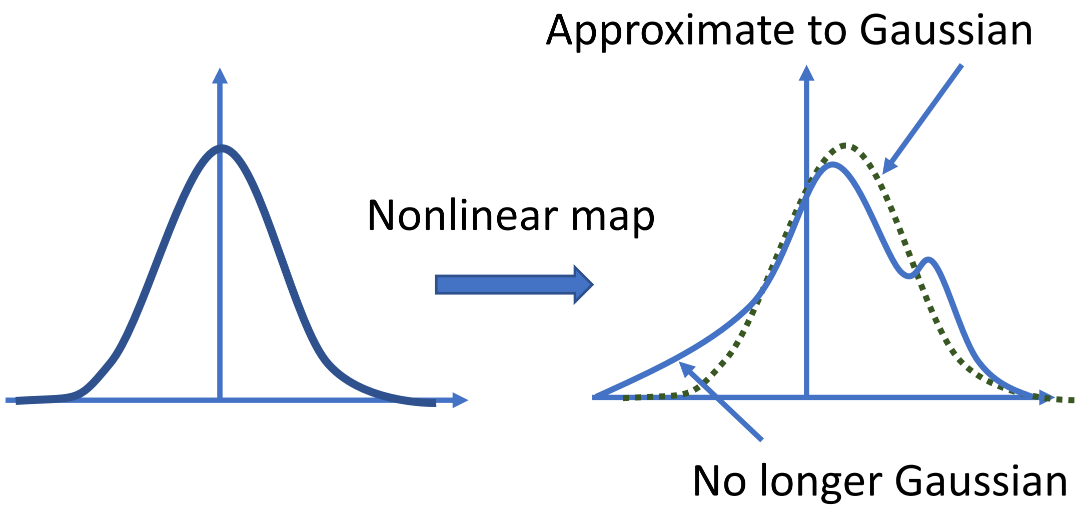
- ❑ Update the state estimate with the Kalman gain;

$$\hat{x}_t = \hat{x}_{t|t-1,sample} + K_t[y_t - \hat{y}_{t,sample}]$$

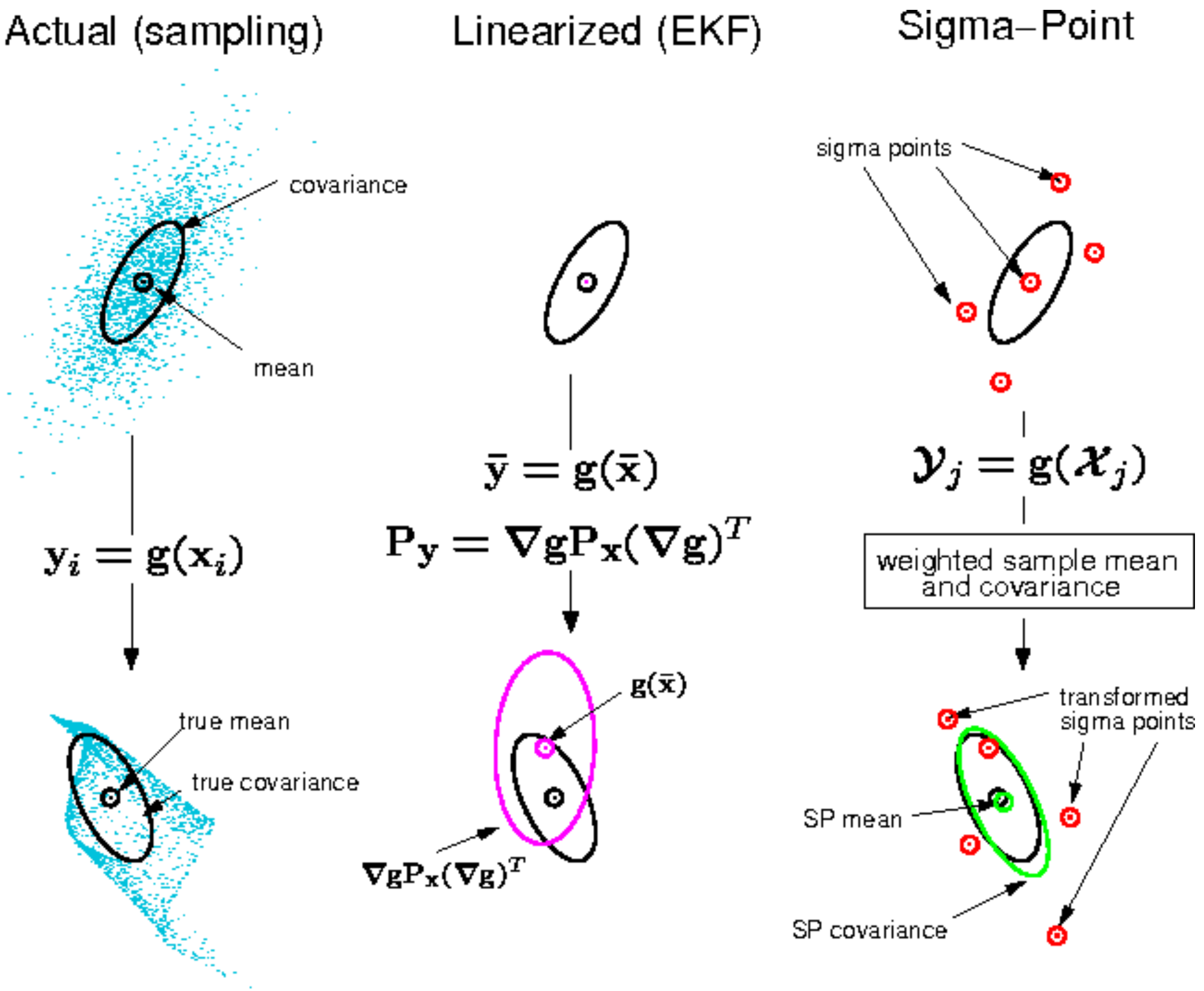
- ❑ Update the a posteriori covariance;

$$P_t \cong P_{t|t-1,sample} - K_t P_y K_t^T$$

- ❑ Set $t = t + 1$, and repeat the above process.



- ❑ No Jacobian, no partial derivatives are needed.
- ❑ The estimated covariance using sigma points is more accurate than the Jacobian-based one.
- ❑ ***Caveat!*** The distribution of random variables after transformed through nonlinear equations, e.g. $f(x,t)$, $h(x,t)$, is no longer Gaussian, although the original distribution was Gaussian. Unscented Kalman Filter approximates this distribution to a Gaussian and characterizes with mean and covariance. Although this approximation is accurate to the 2nd order, the discrepancy from a complete Gaussian may grow, as the process is repeated.



- EKF tends to underestimate covariance P_t and thereby uses a smaller Kalman gain.
- This leads to an insufficient correction (update) to the state estimation.
- Such an inaccurate state estimation further incurs inaccurate covariance and the situation may smallball, leading to a divergence.
- UKF is more reliable particularly when covariances rapidly change.

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