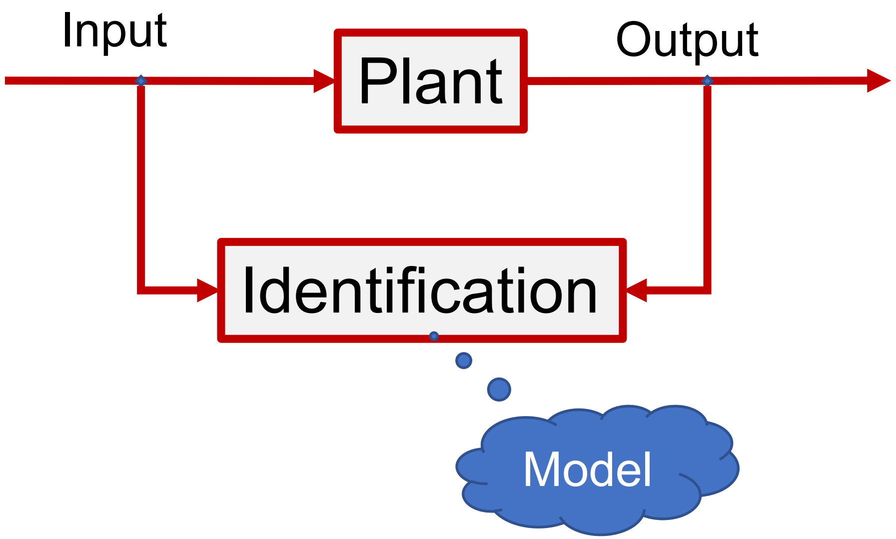
2.160 Identification, Estimation, and Learning

Part 3 Linear System Identification

Lecture 13

Non-Parametric Linear System Identification



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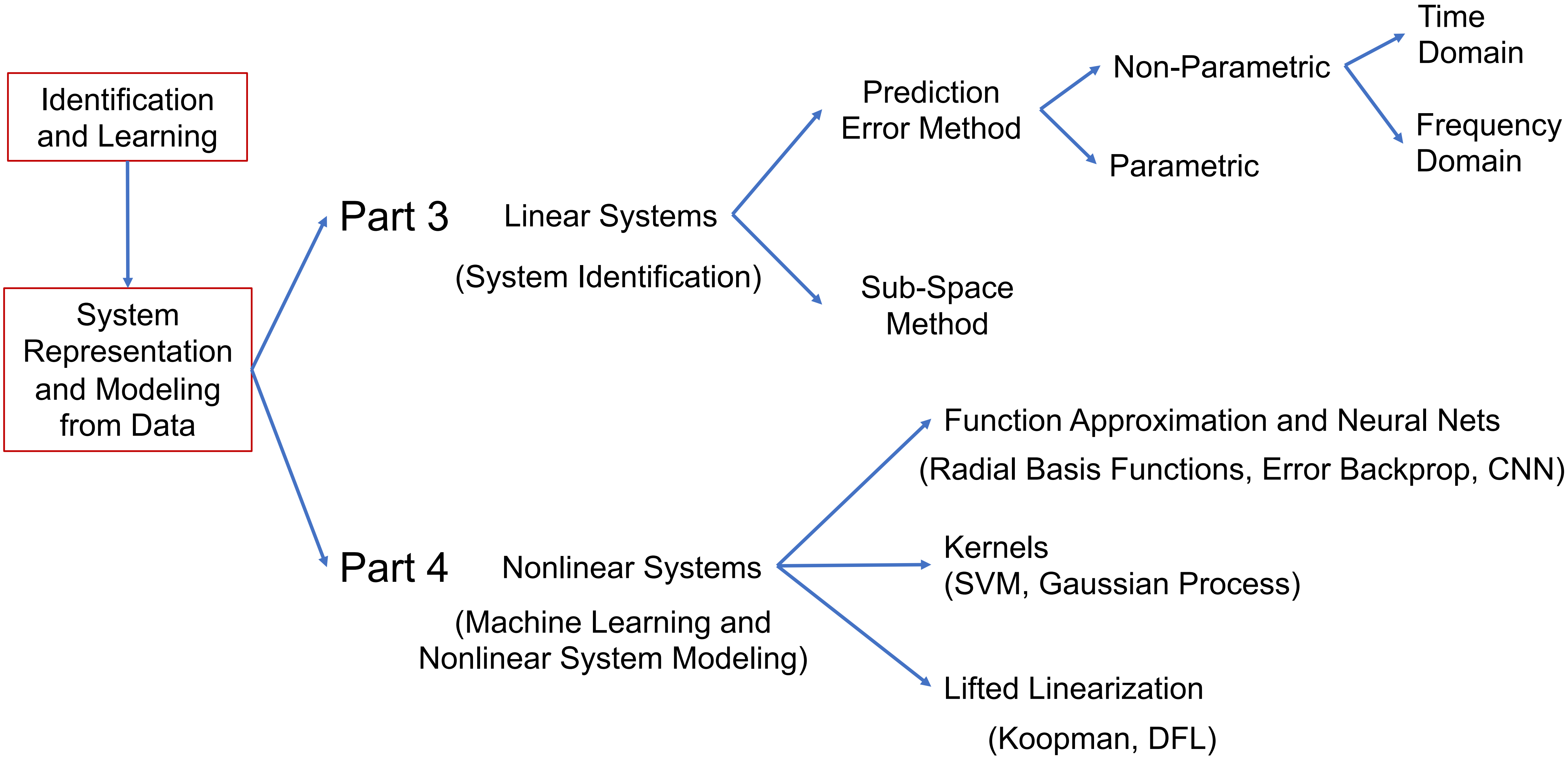
The Second Half of the Course



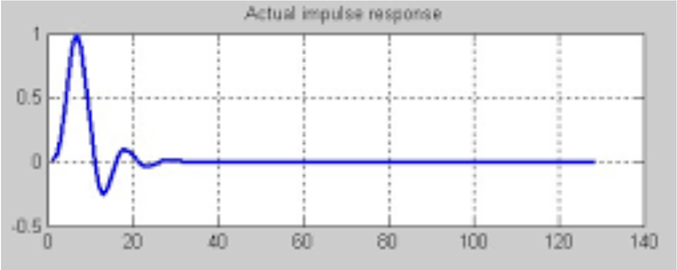
Non-Parametric Representation

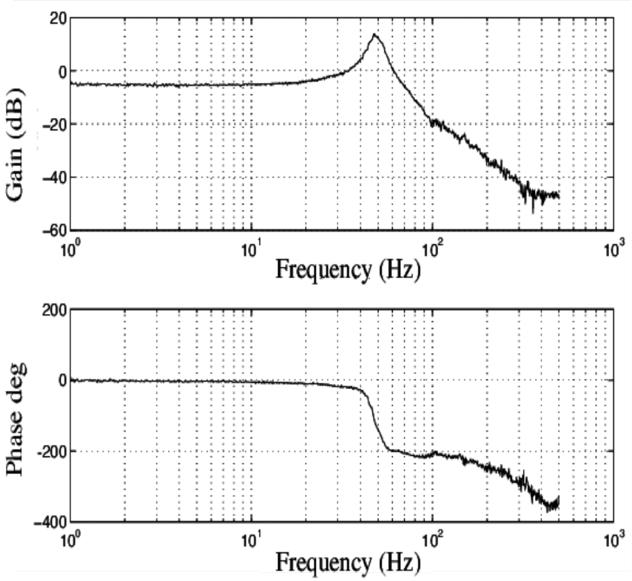
Time Domain

Frequency Domain

Bode Plot, Nyquist Plot

Impulse Response





Parametric Representation

Transfer Function

$$G(s) = \frac{b_1 s + b_2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

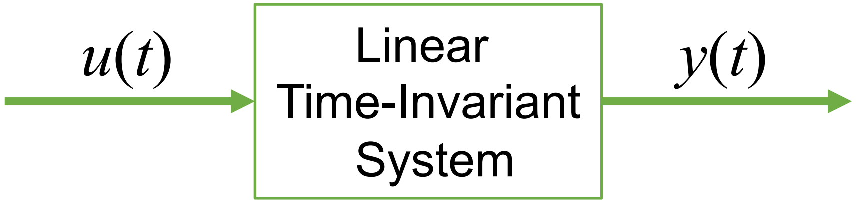
State Equation

$$\dot{x} = Ax + Bu$$

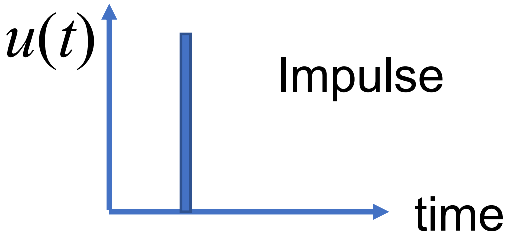
$$y = Cx$$

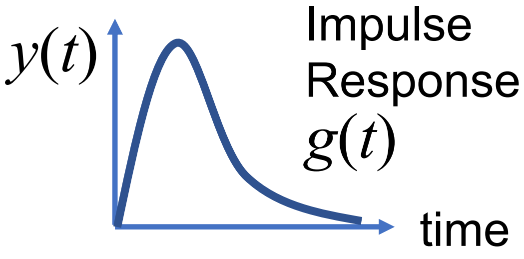
10. Non-Parametric Identification of Linear Time-Invariant Systems

Review



Input

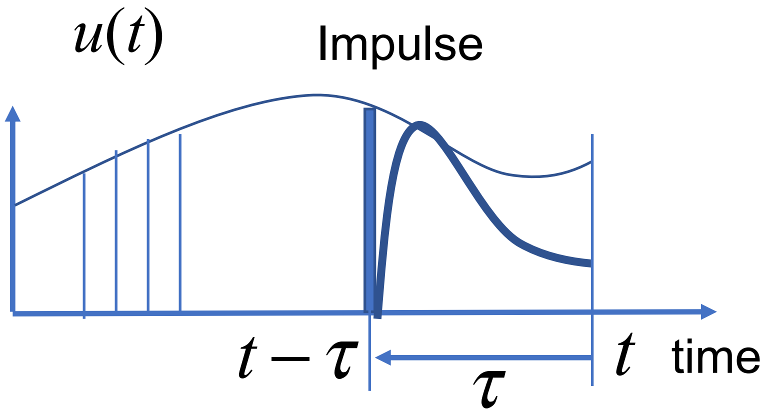




Convolution

$$y(t) = \int_0^{\infty} g(\tau)u(t-\tau)\,d\tau$$

Output $y(t)$ can be expressed as superposition of the effect of all the impulses given to the system until time t.



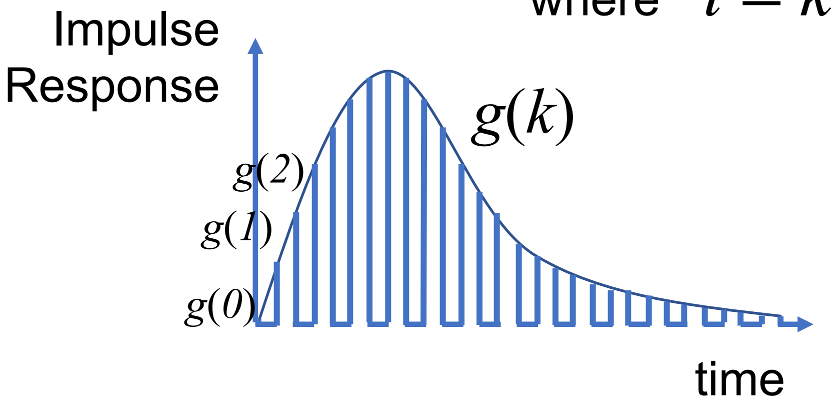
An arbitrary linear time-invariant (LTI) system can be completely characterized with an impulse response: $g(t)$.

Discrete-Time Impulse Response

- We will mainly deal with discrete-time systems with impulse response:

$$\{g(k) \mid k = 0,1,2,3,\cdots\}$$

where $t = k\Delta t$



- For brevity, we assume Δt = 1, and use t and k in an interchangeable manner.

Convolution

$$y(t) = \sum_{k=0}^{\infty} g(k)u(t-k)$$

- **Time-Shift Operator**, q, advances the time index one unit time ahead:

$$qu(t) = u(t+1) \qquad q^2u(t) = u(t+2)$$

$$q^{-1}u(t) = u(t-1)$$

- Using this time-shift operator, we can write the impulse response

$$y(t) = \sum_{k=0}^{\infty} g(k)u(t-k)$$

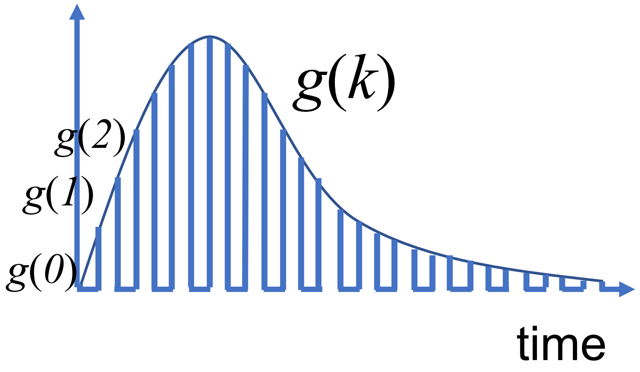
$$= \sum_{k=0}^{\infty} g(k)q^{-k}u(t) = G(q)u(t)$$

where $G(q) = \sum_{k=0}^{\infty} g(k)q^{-k}$

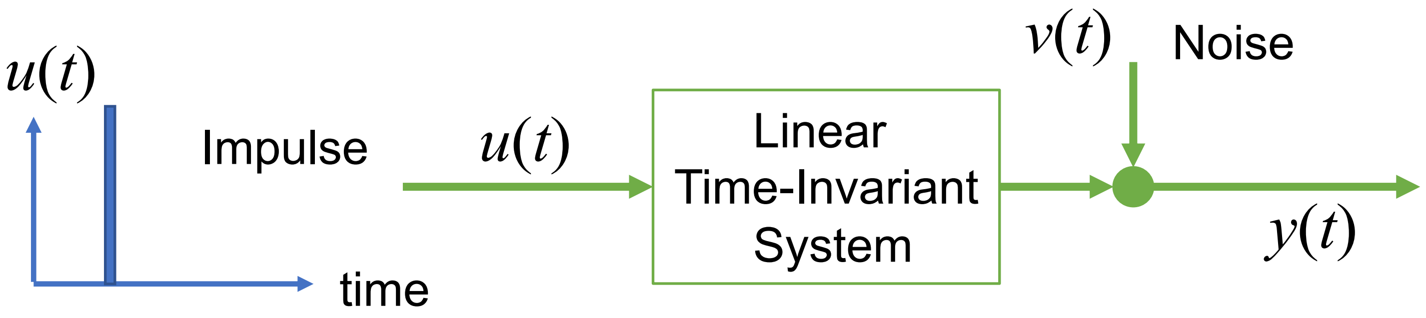
- The impulse response is represented as a polynomial of time shift operator q. We call $G(q)$ Transfer Operator or Transfer Function.

Impulse Response Test

- Impulse Response Test is a simple method for identifying a Linear Time-Invariant (LTI) system, i.e. determine the coefficients $\{g(k)|k=1,2,3\ldots\}$ or $G(q)$, by applying an impulse input and observing the output.
- A challenge is to reduce the effect of noise or disturbance acting on the system.



- A Naïve Method



$$y(t)=\sum_{k=0}^{\infty} g(k)u(t-k)+v(t)$$

$$u(t-k)=\begin{cases} A, & t=k \\ 0, & t\neq k \end{cases}$$

$$y(t)=Ag(t)+v(t)$$

$$\hat{g}(t)=\frac{1}{A}y(t)=g(t)+\frac{1}{A}v(t)$$

$$t=0,1,2,\cdots$$

- The impulse response can be identified by dividing the observed output by the input magnitude A.
- However, it comes with the noise attenuated by the input amplitude.
- To reduce the effect of the noise, the input amplitude must be increased, but there is a physical limitation.

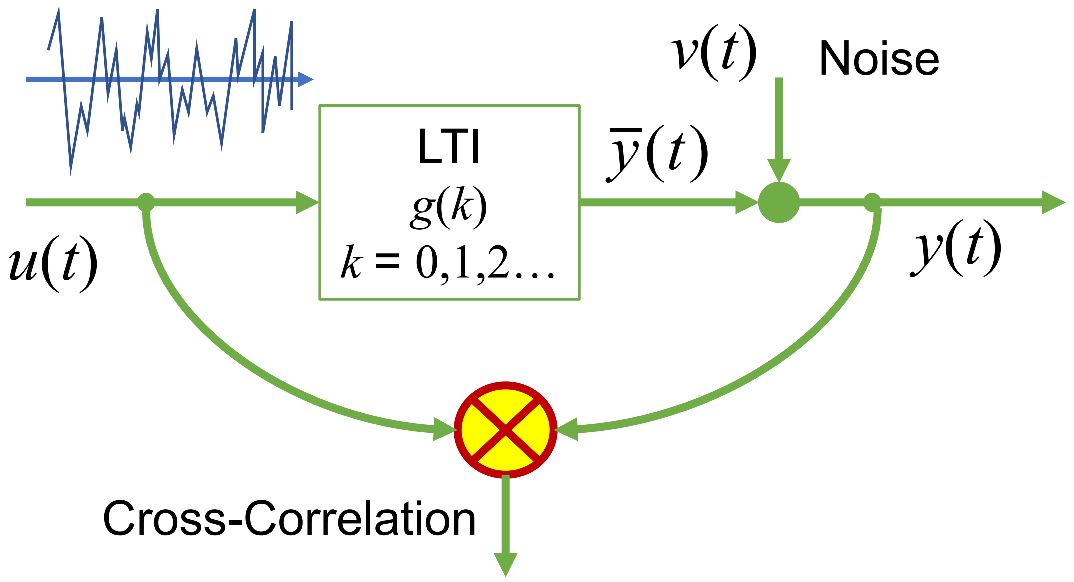
Correlation Method for Identifying Impulse Response Coefficients

- The naïve impulse response test cannot produce a reliable result when the data are corrupted with noise.

- A better alternative is ***Correlation Method***, which allows us to eliminate the effect of noise.

- Idea: Apply an input sequence $u(t)$ that is uncorrelated with noise $v(t)$ to a linear time-invariant system to identify. When taking input-output cross-correlation, the noise that is uncorrelated with the input can be eliminated and, thereby, the system can be identified without being affected by noise.

- The input sequence can be a random signal that is uncorrelated (white).



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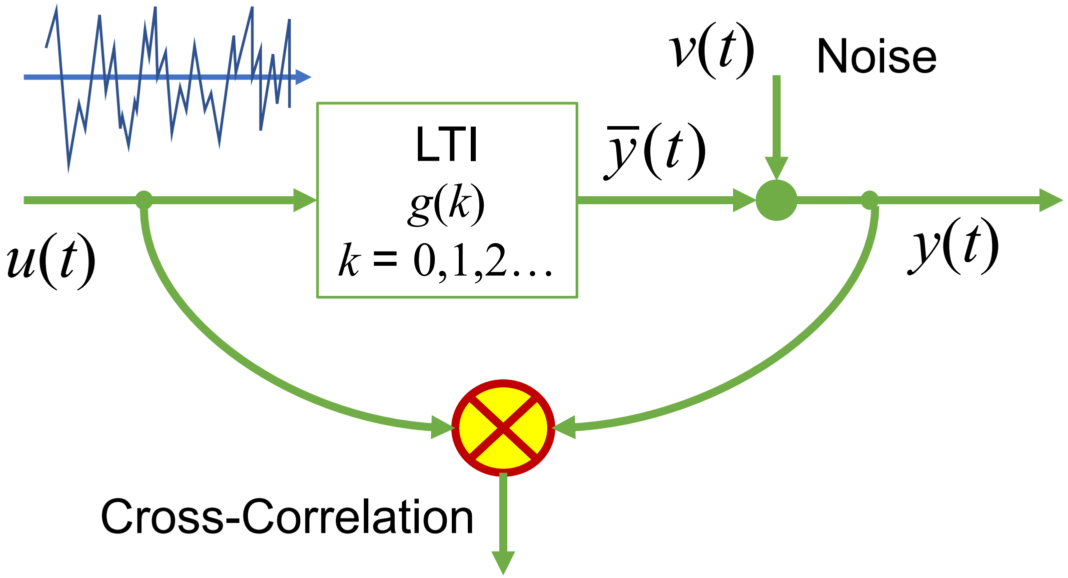


- Let $\bar{y}$ be the noise-free output of the system. Namely, the measured noise-corrupted output is written as

$$y(t) = \bar{y}(t) + v(t)$$

- The input-output cross-correlation is given by

$$\begin{aligned} R_{uy}(\tau) &= E[u(t)y(t+\tau)] \\ &= E[u(t)\bar{y}(t+\tau)] + \cancel{E[u(t)v(t+\tau)]} \\ &= E[u(t)\bar{y}(t+\tau)] \end{aligned}$$

0, Note that the noise is uncorrelated with the input sequence.

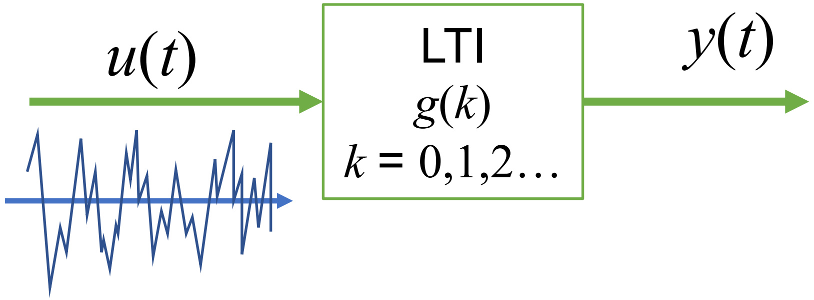
- We need the Wiener-Hopf Equation to obtain the coefficients of impulse response from the above expression.

The Wiener-Hopf Equation

- The Wiener-Hopf Equation provides the theoretical foundation for the correlation method.
- Consider a time-sequence input $u(t)$, which is assumed to be wide-sense stationary.
- Recall that, in a wide-sense stationary process,
  - The mean does not vary over time; and
  - Auto correlation exists and it does not depend on time.

$$R_u(\tau) = E[u(t)u(t+\tau)]$$

- We also assume ergodicity, that is, ensemble mean is equal to time average. Therefore, the autocorrelation can be written as



$$R_u(\tau) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{t=-N}^{N} u(t)u(t+\tau)$$

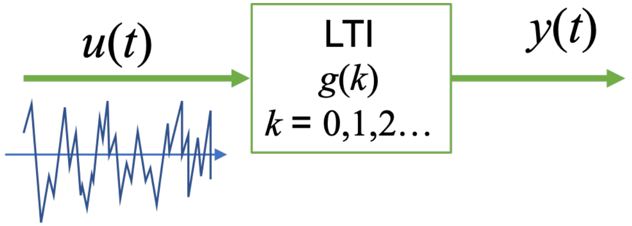
- Suppose that the input sequence is given to an Linear Time Invariant (LIT) system with impulse response;

$$\{g(k) \mid k = 0,1,2,3,\cdots\}$$

- Consider the cross-correlation between the input and the output:

$$R_{uy}(\tau) = E[u(t)y(t+\tau)]$$

The Wiener-Hopf Equation



- This cross-correlation can be written as

$$R_{uy}(\tau) = E[u(t)y(t+\tau)] = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{t=-N}^{N} u(t)y(t+\tau)$$

$$y(t+\tau) = \sum_{k=0}^{\infty} g(\tau)u(t+\tau-k)$$

- Substituting the convolution equation into $y(t+\tau)$ and swapping the two summations yield

$$R_{uy}(\tau) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{t=-N}^{N} u(t) \sum_{k=0}^{\infty} g(k)u(t+\tau-k)$$

$$= \sum_{k=0}^{\infty} \left[\lim_{N\to\infty} \frac{1}{2N+1} \sum_{t=-N}^{N} u(t)u(t+\tau-k) \right] g(k) = \sum_{k=0}^{\infty} R_u(\tau-k)g(k)$$

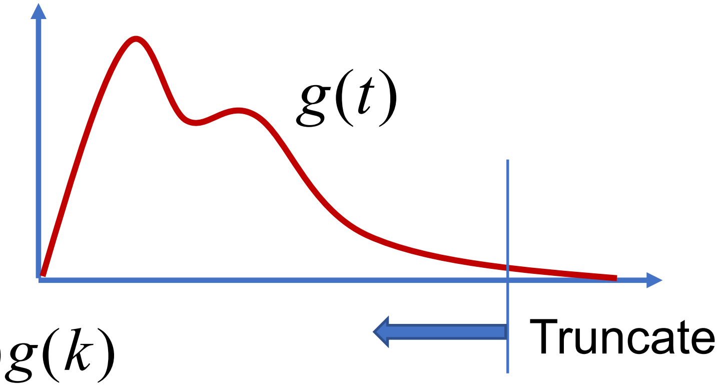
- Therefore, the input-output Cross-Correlation is equal to the convolution of the input autocorrelation with the impulse response of the linear time invariant system. This is the ***Wiener-Hopf Equation***.

$$\therefore R_{uy}(\tau) = \sum_{k=0}^{\infty} R_u(\tau-k)g(k) \quad \longleftrightarrow \quad y(t) = \sum_{\tau=0}^{\infty} u(t-\tau)g(\tau)$$

Analogy

The Wiener-Hopf Equation and the Correlation Method for System Identification

- Suppose that the LTI system to identify is stable. Then, $\lim_{t\to\infty} g(t) = 0$
- The impulse response coefficients can be truncated:
  Finite Impulse Response,



$$\{g(k) \mid k = 0,1,2,\cdots,N\}$$

- The Wiener-Hopf Equation, too, can be truncated: $R_{uy}(\tau) = \sum_{k=0}^{N} R_u(\tau - k) g(k)$
- This can be written for $0 \le \tau \le N$.

$$\tau = 0: \; R_{uy}(0) = R_u(0)g(0) + R_u(-1)g(1) + \cdots + R_u(-N)g(N)$$

$$\vdots$$

$$\tau = N: R_{uy}(N) = R_u(N)g(0) + R_u(N-1)g(1) + \cdots + R_u(0)g(N)$$

Or, in vector and matrix form

$$\begin{pmatrix} R_{uy}(0) \\ R_{uy}(1) \\ \vdots \\ R_{uy}(N) \end{pmatrix} = \underbrace{\begin{pmatrix} R_u(0) & R_u(-1) & \cdots & R_u(-N) \\ R_u(1) & R_u(0) & & \vdots \\ \vdots & & \ddots & \vdots \\ R_u(N) & \cdots & \cdots & R_u(0) \end{pmatrix}}_{\boldsymbol{R}_N} \begin{pmatrix} g(0) \\ g(1) \\ \vdots \\ g(N) \end{pmatrix}$$

The Wiener-Hopf Equation and the Correlation Method for System Identification

- Assuming that the above matrix $\boldsymbol{R}_N$ is non-singular, we can obtain the coefficients of impulse response.

$$\begin{pmatrix} g(0) \\ g(1) \\ \vdots \\ g(N) \end{pmatrix} = \boldsymbol{R}_N^{-1} \begin{pmatrix} R_{uy}(0) \\ R_{uy}(1) \\ \vdots \\ R_{uy}(N) \end{pmatrix}$$

- Consider a random signal that is uncorrelated (white), as mentioned previously.

$$R_u(\tau) = E[u(t)u(t+\tau)] = \begin{cases} \lambda: & \tau = 0 \\ 0: & \tau \neq 0 \end{cases}$$

$$\boldsymbol{R}_N = \begin{pmatrix} R_u(0) & R_u(-1) & \cdots & R_u(-N) \\ R_u(1) & R_u(0) & & \vdots \\ \vdots & & \ddots & \vdots \\ R_u(N) & \cdots & \cdots & R_u(0) \end{pmatrix}$$

- The matrix $\boldsymbol{R}_N$ then becomes diagonal.

$$\boldsymbol{R}_N = \begin{pmatrix} \lambda & & & 0 \\ & \lambda & & \\ & & \ddots & \\ 0 & & & \lambda \end{pmatrix}$$

- Therefore, the impulse response coefficients are

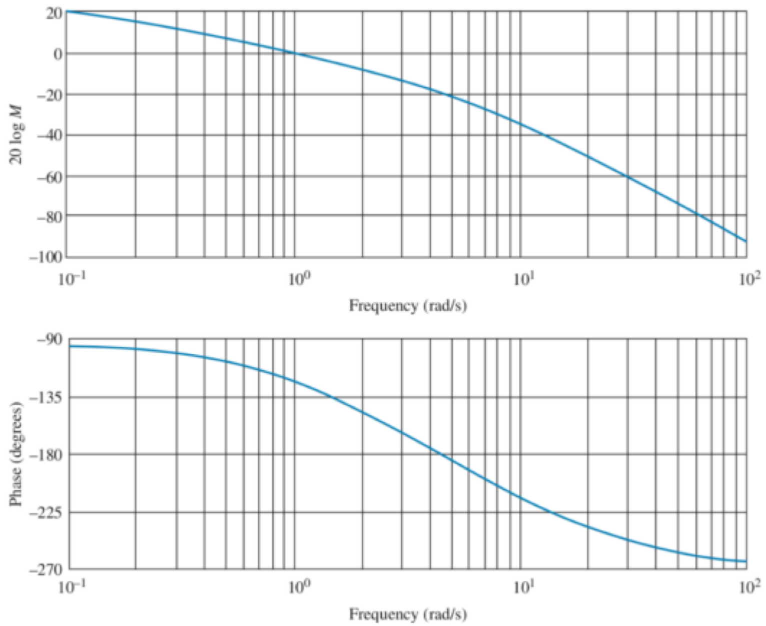
$$\hat{g}(t) = \frac{1}{\lambda} R_{uy}(t), \quad t = 0, 1, \cdots, N$$

- Note that the input sequence is generated such that it does not correlate with noise. Therefore, it is eliminated, as shown previously. This is the ***Correlation Method*** for identifying impulse response coefficients.

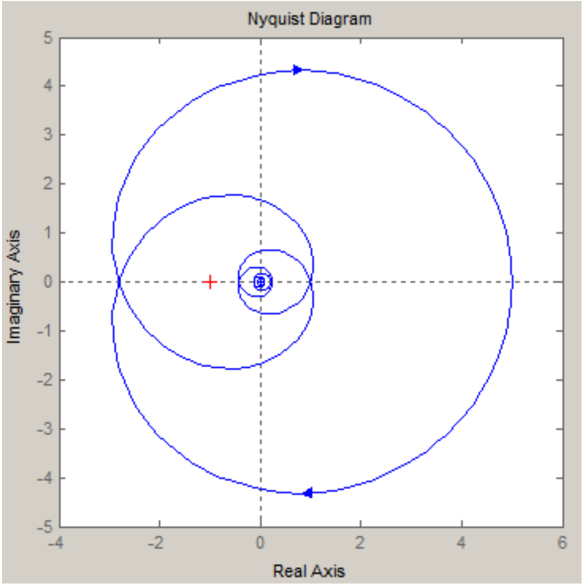
- In practice, completely noise free identification is infeasible, because it is infeasible to generate an input sequence that is perfectly uncorrelated with noise.

A Frequency-Domain Approach to Non-Parametric System Identification

- The correlation method for identifying coefficients of impulse response is a time-domain approach to identifying linear time-invariant systems. The method is underpinned by the Wiener-Hopf Equation.
- While impulse response is a time-domain representation of LTI systems, there is another representation of LTI systems, that is the one based on Frequency Response. Bode plot and Nyquist plot are examples of the frequency-domain graphical representation.
- The frequency-domain approach has been widely used in industry. It is practical and robust, having many effective tools underpinned by important theories and techniques.
- To learn the frequency-domain approach, we need to introduce, or review,
  - Discrete-Time Fourier Transform,
  - Power Spectrum and Cross-Spectrum,
  - Frequency Transfer Function, and
  - Coherence.



Bode Plot

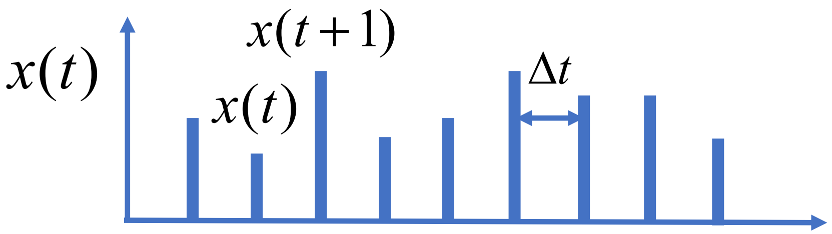


Nyquist Plot

Discrete-Time Fourier Transform

- Consider a time sequence

$$\{\cdots, x(t-1), x(t), x(t+1), \cdots\}$$



- Discrete-Time Fourier Transform (DTFT) of $\{x(t)\}$ is given by

$$X(\omega) \triangleq \sum_{k=-\infty}^{\infty} x(k)\exp(-i\omega k)$$

where $\exp(-i\omega k) = \cos\omega k + i\sin\omega k$

- Note that $X(\omega)$ is a complex function of ω;
- $X(\omega)$ is a periodic function:

$$X(\omega+2\pi) = X(\omega), \quad -\pi \le \omega < \pi$$

- Caveat! For brevity we have set Δt to 1, but to obtain the real, physical frequency, we must convert the frequency ω used in DTFT to the real, physical frequency:

$$t_{real} = \Delta t \cdot k \quad \omega_{real} t_{real} = \underbrace{\omega_{real}\Delta t}_{\omega} \cdot k \quad \therefore \omega_{real} = \frac{\omega}{\Delta t}$$

Example: If Δt = 1 ms, then $\omega_{real} = \omega \times 10^3 \, rad/s$

$$-\pi \le \omega < \pi \quad \rightarrow \quad -10^3\pi \le \omega_{real} < 10^3\pi$$

- Inverse Transform

$$\int_{-\pi}^{\pi} X(\omega)\exp(i\omega k)d\omega = \int_{-\pi}^{\pi} \sum_{\ell=-\infty}^{\infty} x(\ell)\exp(-i\omega\ell)\exp(i\omega k)d\omega$$

$$= \sum_{\ell=-\infty}^{\infty} x(\ell)\int_{-\pi}^{\pi} \exp(i\omega(k-\ell))\,d\omega = 2\pi x(k)$$

$$\begin{cases} \ell \ne k: & 0 \\ \ell = k: & 2\pi \end{cases}$$

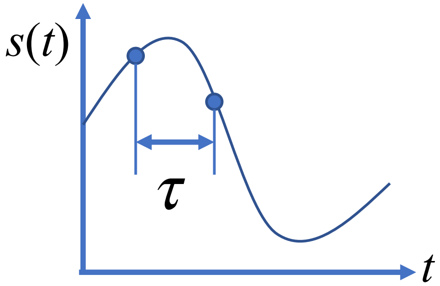
$$\boxed{\therefore x(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\exp(i\omega k)d\omega}$$

- Inverse DTFT completely recovers the original $\{x(t)\}$.

Power Spectrum

- Power Spectrum is the Fourier Transform of Auto-Correlation.
- Consider a wide-sense stationary sequence $\{s(t)\}$, for which the following auto-correlation exists.

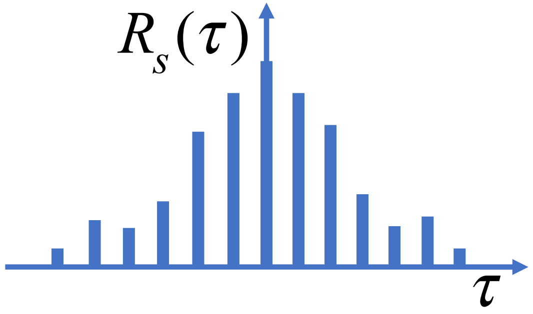
$$R_s(\tau) = E[s(t)s(t+\tau)]$$



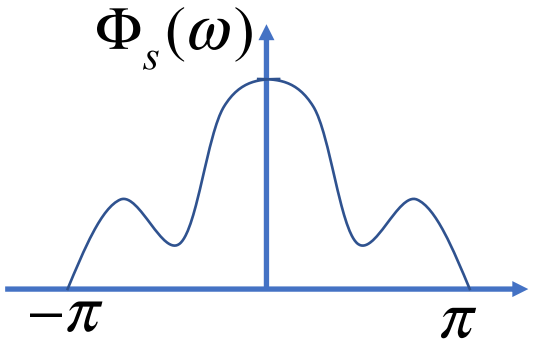
- Taking DTFT, we can obtain its power spectrum:

$$\Phi_s(\omega) = \sum_{\tau=-\infty}^{\infty} R_s(\tau)\exp(-i\omega\tau)$$

Auto-Correlation



Power Spectrum

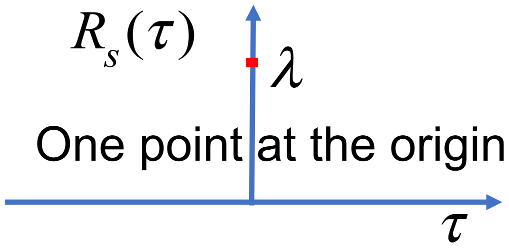


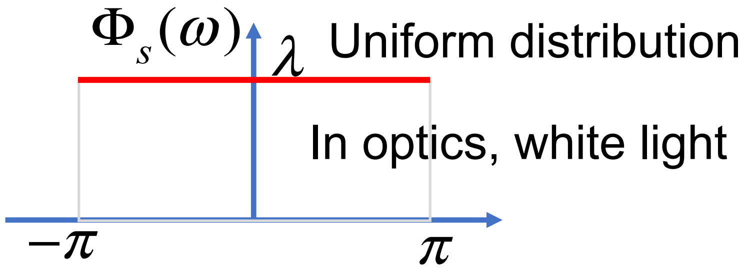
- White Noise: Uncorrelated noise is called White noise, because its power spectrum is uniform, i.e. all the frequency components are equally involved.

$$R_e(\tau) = E[e(t)e(t+\tau)] = \begin{cases} \lambda; & \tau = 0 \\ 0; & \tau \neq 0 \end{cases}$$

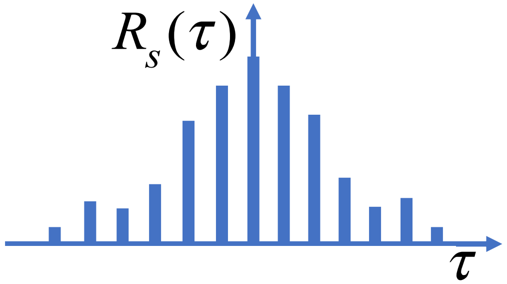
- Power Spectrum of White (uncorrelated) noise

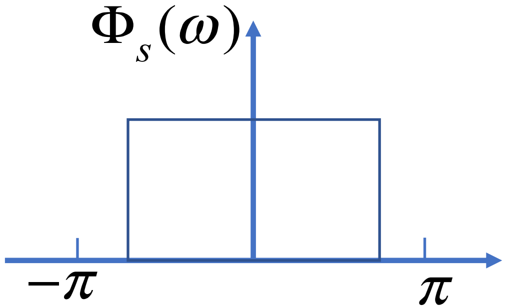
$$\Phi_e(\omega) = \sum_{\tau=-\infty}^{\infty} R_e(\tau)\exp(-i\omega\tau) = \lambda e^0 = \lambda$$





- Band-limited White noise





Ideal Low Pass filtered White Noise

$$S_N(f) = \begin{cases} \frac{N_0}{2}, & -B < f < B \\ 0, & |f| > B \end{cases}$$

$$R_N(\tau) = \int_{-B}^{B} \frac{N_0}{2} \exp(j2\pi f\tau)\, df = N_0 B \,\mathrm{sinc}(2B\tau)$$

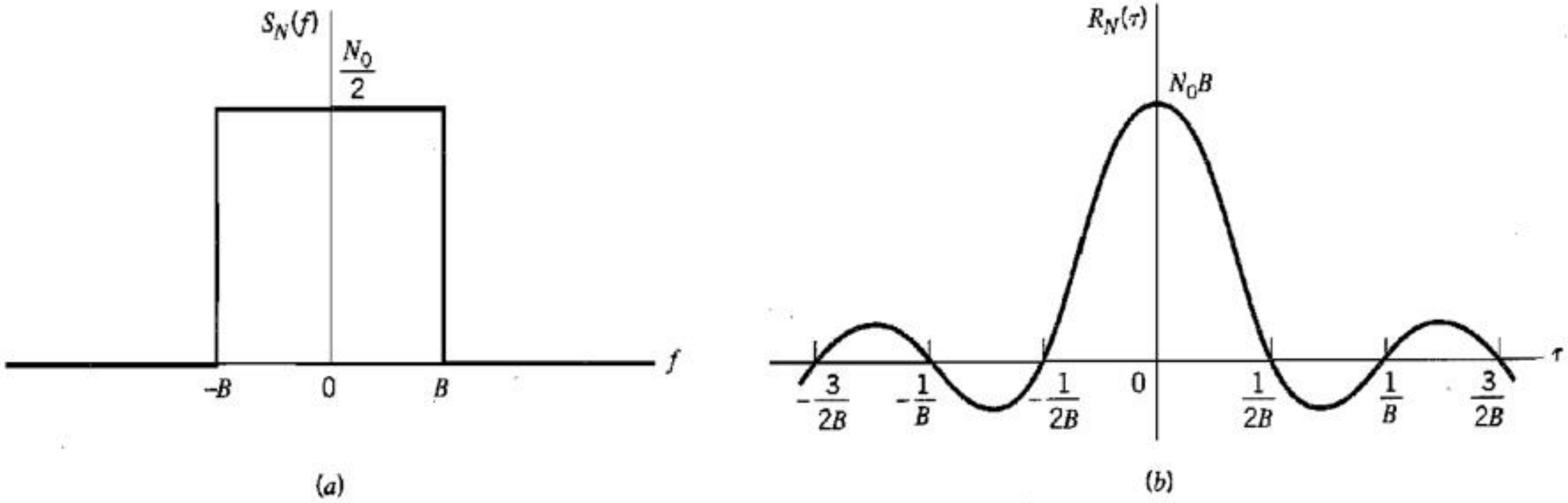


FIGURE 1.17 Characteristics of low-pass filtered white noise. (*a*) Power spectral density. (*b*) Autocorrelation function.

Frequency Transfer Function and Cross-Spectrum

- Recall that, given a transfer function $G(s)$ obtained from a differential equation in continuous time through Laplace transform, we can find its frequency transfer function $G(i\omega)$ by replacing s by $i\omega$.

- Likewise, given a discrete-time transfer operator (function) $G(q)$, its frequency transfer function can be obtained by replacing q by $e^{i\omega}$; $G(e^{i\omega})$.

- Let $R_{uy}(\tau)$ be the cross-correlation from input u to output y of a LTI system. The Cross-Spectrum from input to output is the discrete-time Fourier Transform of the input-output cross-correlation.

$$\Phi_{uy}(\omega) = \sum_{\tau=-\infty}^{\infty} R_{uy}(\tau)\exp(-i\omega\tau)$$

- We can show that the frequency transfer function is given by the cross-spectrum divided by the input power spectrum.

$$G(e^{i\omega}) = \frac{\Phi_{uy}(\omega)}{\Phi_u(\omega)}$$

Proof of $G(e^{i\omega}) = \dfrac{\Phi_{uy}(\omega)}{\Phi_u(\omega)}$

- Consider a LTI system with impulse response coefficients $g(k), k = 0,1,\cdots$.
- Recall the Wiener-Hopf Equation relating the input-output cross-correlation $R_{uy}(\tau)$ to input autocorrelation $R_u(\tau)$

$$R_{uy}(\tau) = \sum_{k=0}^{\infty} g(k) R_u(\tau - k)$$

where the input sequence is assumed to be wide-sense stationary.

- Taking Discrete-Time Fourier Transform of the cross-correlation and using the Wiener-Hopf equation yield,

$$\Phi_{uy}(\omega) = \sum_{\tau=-\infty}^{\infty} R_{uy}(\tau)\exp(-i\omega\tau) = \sum_{\tau=-\infty}^{\infty}\sum_{k=0}^{\infty} g(k)R_u(\tau-k)\exp(-i\omega\tau) \leftarrow \textit{Multiplying } \exp(-i\omega k)\exp(i\omega k)$$

$$= \sum_{k=0}^{\infty} g(k)\exp(-i\omega k)\sum_{\tau=-\infty}^{\infty} R_u(\underbrace{\tau-k}_{s})\exp(-i\omega(\underbrace{\tau-k}_{s})) = \boxed{\sum_{k=0}^{\infty} g(k)\exp(-i\omega k)}\sum_{s=-\infty}^{\infty} R_u(s)\exp(-i\omega s)$$

$$= G(e^{i\omega})\cdot\Phi_u(\omega)$$

Recall $G(q) = \sum_{k=0}^{\infty} g(k)q^{-k} \rightarrow G(e^{i\omega})$

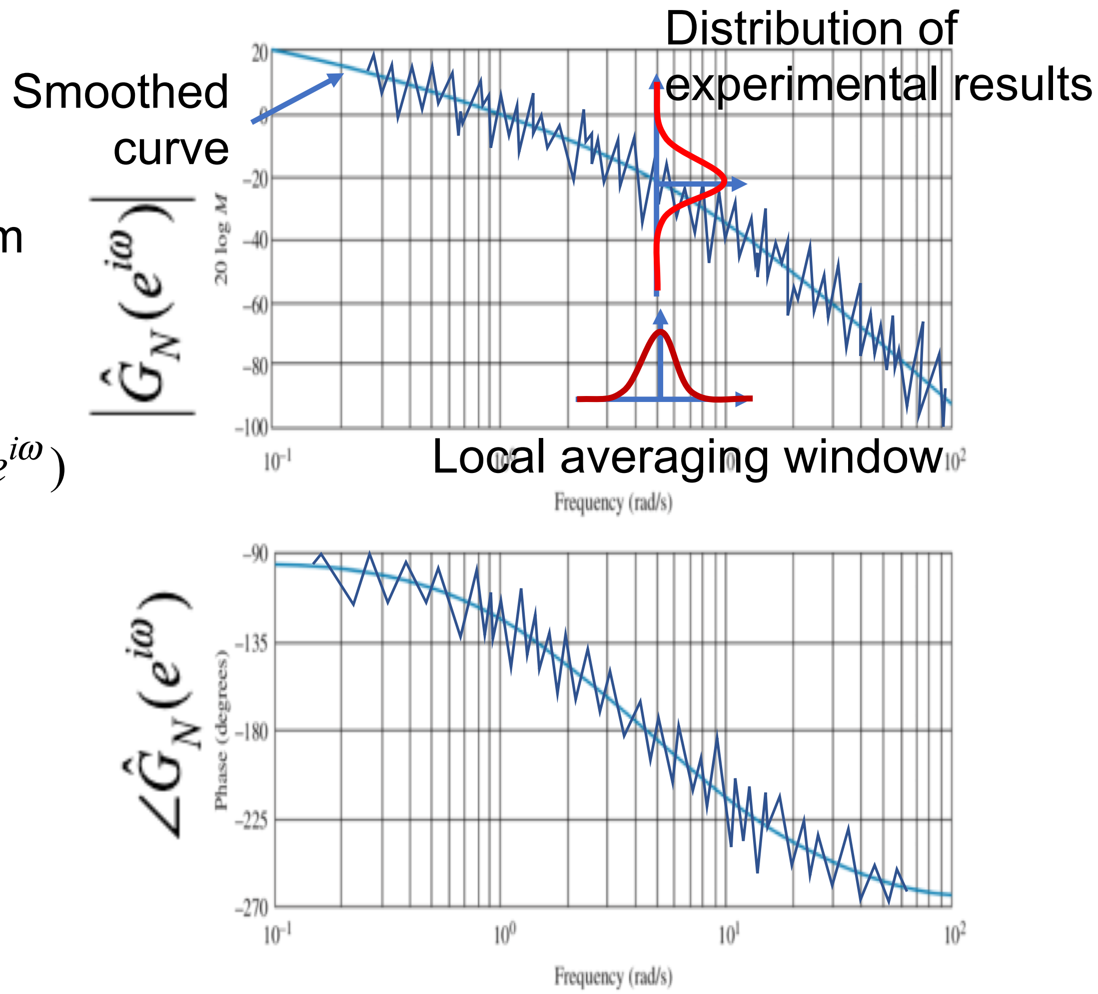
$$\therefore G(e^{i\omega}) = \frac{\Phi_{uy}(\omega)}{\Phi_u(\omega)}$$

Bode Plot

- A Bode Plot can be generated based on the above formula: $G(e^{i\omega}) = \dfrac{(\text{Cross Spectrum})}{(\text{Power Spectrum})}$
- Suppose that we have input output data

$$\{(u(t), y(t)) \mid t = 1, 2, \cdots, N\}$$

- Compute auto-correlation of the input and the cross-correlation with the output, and then compute power spectrum and cross-spectrum.
- Note that cross spectrum is a complex function, and thereby the frequency transfer function is a complex function of frequency. Computing the magnitude and phase angle of $G(e^{i\omega})$ yields the magnitude and phase diagrams of Bode plot.
- It is known that as the number of data tends to infinity, the estimated transfer function converges to the true transfer function <u>in average</u>: $\hat{G}_N(e^{i\omega}) \underset{N\to\infty}{\longrightarrow} G_0(e^{i\omega})$
- However, its variance does not converge to zero.
- Experimentally obtained Bode plots are, in general, not smooth. The jagged Bode plot can be smoothed out by using a local averaging technique.
- Hamming Window is widely used for smoothing: Spectro Analyzer.



Properties of Power Spectrum, Cross Spectrum, and Frequency Transfer Function

- Power spectrum $\Phi_u(\omega)$ is a real even function.

$$\Phi_u(-\omega) = \Phi_u(\omega) \qquad \Phi_u(\omega) = \sum_{\tau=-\infty}^{\infty} R_u(\tau)\exp(-i\omega\tau)$$

Recall $\exp(-i\omega k) = \cos\omega k + i\sin\omega k$

$$= R_u(0)e^0 + R_u(1)e^{-i\omega} + R_u(-1)e^{i\omega} + R_u(2)e^{-i2\omega} + R_u(-2)e^{i2\omega} + \cdots$$

Imaginary terms cancel $\quad = R_u(0) + R_u(1)e^{-i\omega} + R_u(1)e^{i\omega} + R_u(2)e^{-i2\omega} + R_u(2)e^{i2\omega} + \cdots$

$$= R_u(0) + R_u(1)\cos\omega + R_u(2)\cos 2\omega + R_u(3)\cos 3\omega + \cdots \quad : \ \text{real}$$

- Cross spectrum $\Phi_{uy}(\omega)$ is a complex, skew-symmetric function.

$$\Phi_{uy}(-\omega) = \Phi_{yu}(\omega) \qquad \Phi_{uy}(-\omega) = \sum_{\tau=-\infty}^{\infty} R_{uy}(\tau)e^{i\omega\tau} = \sum_{\tau'=\infty}^{-\infty} R_{uy}(-\tau')e^{-i\omega\tau'} \leftarrow R_{uy}(\tau) = R_{yu}(-\tau)$$

$$= \sum_{\tau'=-\infty}^{\infty} R_{yu}(\tau')e^{-i\omega\tau'} = \Phi_{yu}(\omega)$$

- If $u(t)$ and $v(t)$ are uncorrelated, the power spectrum of $y = u + v$ is

$$\Phi_y(\omega) = \Phi_u(\omega) + \Phi_v(\omega) \qquad \because R_y(\tau) = R_u(\tau) + R_v(\tau)$$

Properties of Power Spectrum, Cross Spectrum, and Frequency Transfer Function

- The squared magnitude of a frequency transfer function $\left|G(e^{i\omega})\right|^2$ is given by

$$\left|G(e^{i\omega})\right|^2 = G(e^{i\omega})G(e^{-i\omega})$$

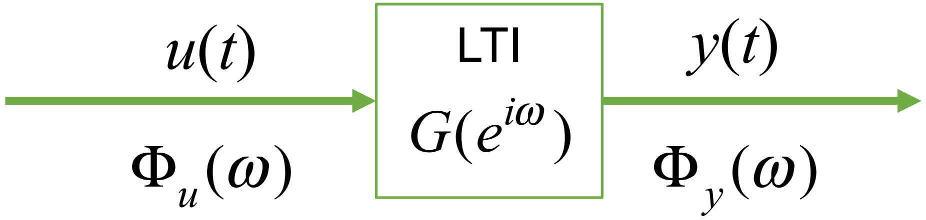
Frequency transfer function is a complex function: $G(e^{i\omega}) = a + ib$

$$G(e^{-i\omega}) = a - ib, \quad G(e^{i\omega})G(e^{-i\omega}) = (a+ib)(a-ib) = a^2 + b^2$$

- For $y(t) = G(q) \cdot u(t)$

$$\Phi_y(\omega) = \left|G(e^{i\omega})\right|^2 \Phi_u(\omega)$$

See the proof in Lecture Notes Chapter 11, pp.4-5.



Coherence

- Coherence is a measure for evaluating the fidelity of identified transfer function $G(e^{i\omega})$.

- Definition:

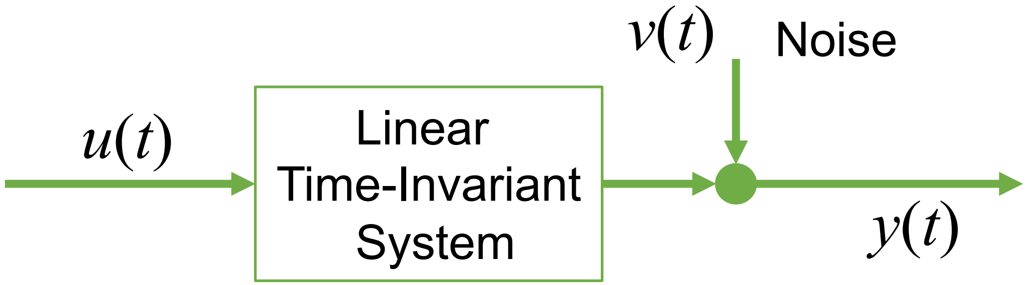
$$\gamma^2(\omega) \triangleq \frac{\left|\Phi_{uy}(\omega)\right|^2}{\Phi_u(\omega)\Phi_y(\omega)}$$

$$0 \le \gamma^2(\omega) \le 1$$

No fidelity High fidelity

- Suppose that noise and/or disturbance $v(t)$ acts on the system as an additive term.

$$y(t) = \bar{y}(t) + v(t) = G(q)\cdot u(t) + v(t)$$



- If $u(t)$ and $v(t)$ are uncorrelated,

$$\Phi_y(\omega) = \Phi_{\bar{y}}(\omega) + \Phi_v(\omega) = \left|G(e^{i\omega})\right|^2 \Phi_u(\omega) + \Phi_v(\omega)$$

- Substituting this into the coherence and noting that $\Phi_{uy}(\omega) = G(e^{i\omega})\Phi_u(\omega)$, and

$$\left|\Phi_{uy}(\omega)\right|^2 = \left|G(e^{i\omega})\right|^2 \left(\Phi_u(\omega)\right)^2$$

$$\gamma^2(\omega) \triangleq \frac{\left|G(e^{i\omega})\right|^2 \Phi_u(\omega)}{\left|G(e^{i\omega})\right|^2 \Phi_u(\omega) + \Phi_v(\omega)} \le 1$$

If no noise and disturbance, then coherence becomes 1.

Coherence

- Low coherence occurs due to
  - Exogenous disturbance and noise
  - Distortion due to nonlinearity
  - Leak in Fast Fourier Transform (F.F.T.)

- Coherence is used for assuring whether the identified transfer function is reliable. It shows a flag when some of the disturbance, nonlinearity, etc. is occurring.

- Coherence also shows in which frequency range it is reliable.