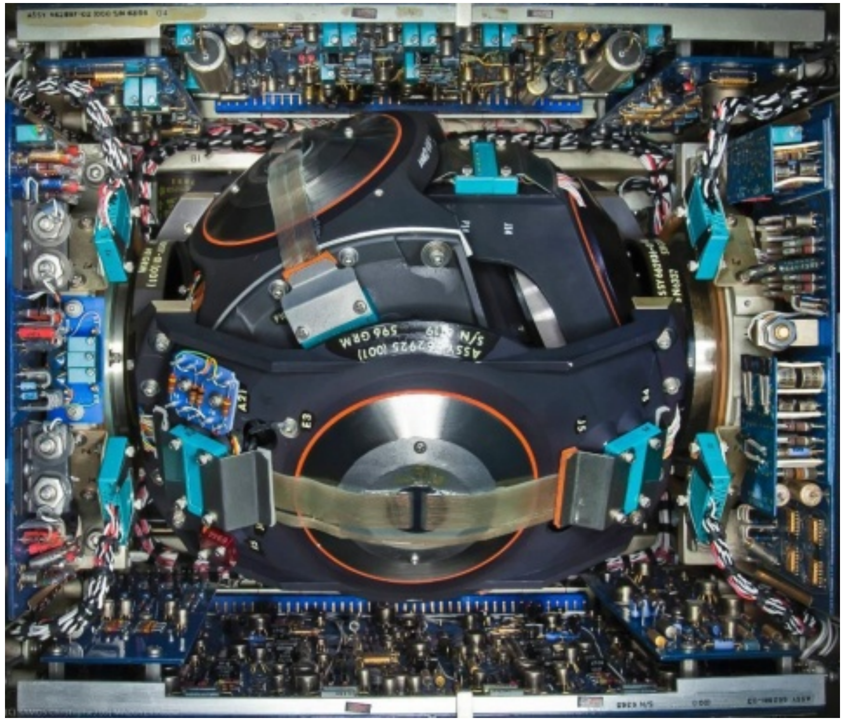
2.160 Identification, Estimation, and Learning

Part 2 Estimation

Lecture 8

Continuous-Time
Kalman Filter



UAV Inertial Navigation with Kalman Filter

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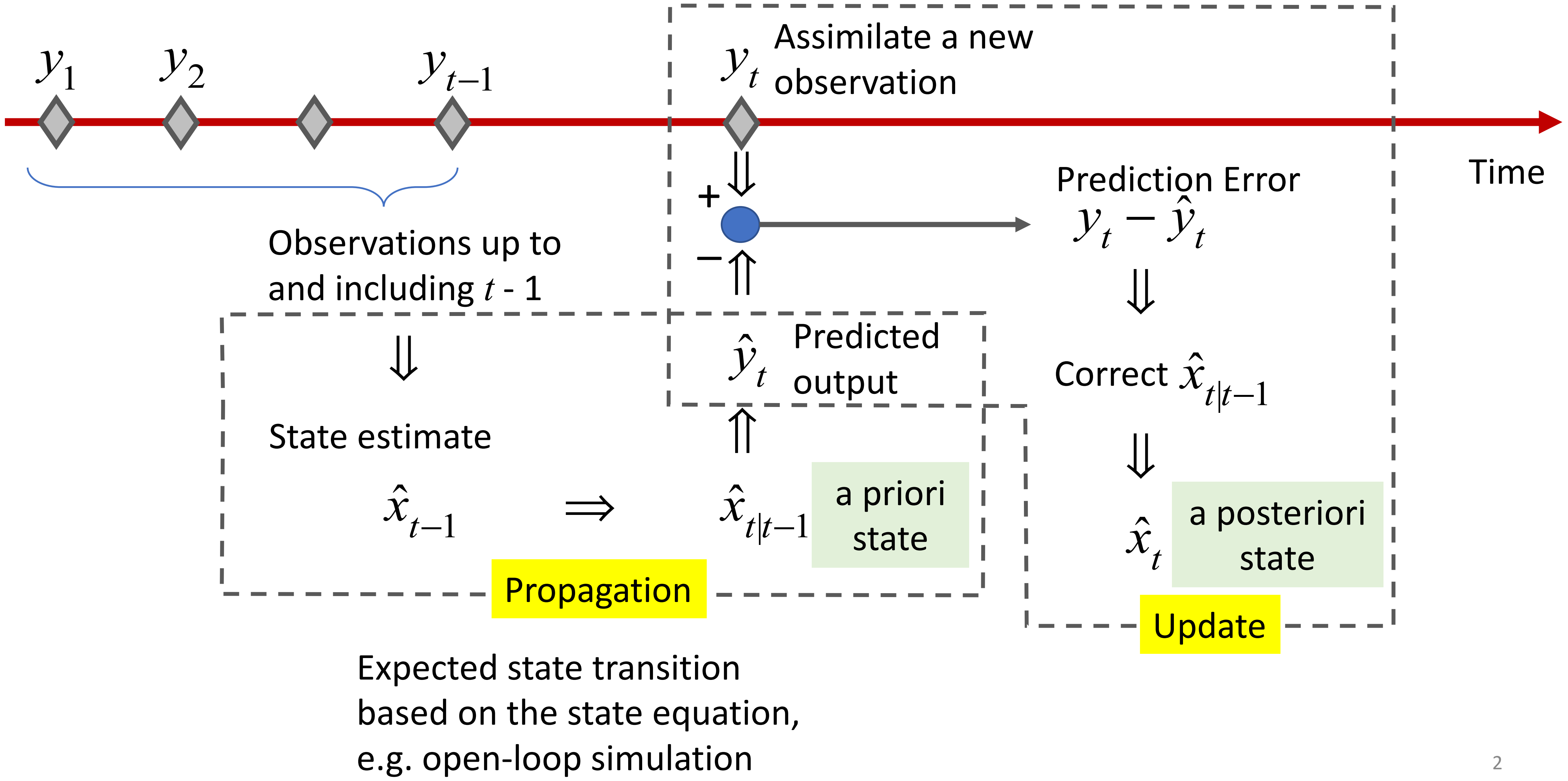
Recap

The Flow of the Discrete Kalman Filter Algorithm



Recursive Formula of Discrete-Time Kalman Filter

x_0 Initial Conditions $P_{1|0}$

State Propagation

$$\hat{x}_{t|t-1} = A_{t-1}\hat{x}_{t-1}$$

$$\hat{y}_t = H_t\hat{x}_{t|t-1} = H_t A_t \hat{x}_{t-1}$$

$$t = t+1$$

$$\hat{x}_t = \hat{x}_{t|t-1} + K_t[y_t - \hat{y}_t]$$

State Update

Covariance Propagation

$$P_{t+1|t} = A_t P_t A_t^T + G_t Q_t G_t^T$$

$$t = t+1$$

$$P_t = (I - K_t H_t) P_{t|t-1}$$

Covariance Update

$$K_t = P_{t|t-1} H_t^T (H_t P_{t|t-1} H_t^T + R_t)^{-1}$$

On the Kalman Gain $K_t = P_{t|t-1} H_t^T (H_t P_{t|t-1} H_t^T + R_t)^{-1}$

- Post-multiplying $H_t P_{t|t-1} H_t^T + R_t$

$$K_t (H_t P_{t|t-1} H_t^T + R_t) = P_{t|t-1} H_t^T \qquad (*)$$

- From the covariance update law $P_t = (I - K_t H_t) P_{t|t-1}$

$$K_t H_t P_{t|t-1} = P_{t|t-1} - P_t$$

- Substituting this into (*)

$$(P_{t|t-1} - P_t) H_t^T + K_t R_t = P_{t|t-1} H_t^T \rightarrow \cancel{P_{t|t-1} H_t^T} - P_t H_t^T + K_t R_t = \cancel{P_{t|t-1} H_t^T}$$

$$\therefore K_t = P_t H_t^T R_t^{-1}$$

- The Kalman gain is proportional to the inverse of the measurement noise covariance R^{-1} and the posteriori prediction error covariance P_t .

Kalman Filter: Continuous v.s. Discrete Time

	Continuous Time	Discrete Time
	Linear Time-Varying System	
State Equation	$\frac{dx}{dt} = F(t)x(t) + G(t)w(t)$	$x_{t+1} = A_t x_t + B_t u_t + G_t w_t$
Measurement Equation	$y(t) = H(t)x(t) + v(t)$	$y_t = H_t x_t + v_t$
State update & propagation	$\frac{d}{dt}\hat{x}(t) = F(t)\hat{x}(t) + K(t)[y(t) - \hat{y}(t)]$	$\hat{x}_t = A_{t-1}\hat{x}_{t-1} + K_t[y_t - \hat{y}_t]$
Covariance update & propagation	The Riccati Differential Equation	$P_t = (I - K_t H_t)P_{t\|t-1}$ $P_{t+1\|t} = A_t P_t A_t^T + G_t Q_t G_t^T$

$$\dot{x} \cong \frac{x_{t+1} - x_t}{\Delta t}$$

$$K_t = P_t H_t^T R_t^{-1}$$

Converting system representation from discrete time to continuous time

State Equation

$$\frac{x_{t+1}-x_t}{\Delta t} \cong F(t)x(t)+G(t)w(t)$$

$$x_{t+1} = \underbrace{(I+F(t)\Delta t)}_{A_t} x_t + \underbrace{G(t)}_{G_t}\underbrace{\Delta t\, w(t)}_{w_t}$$

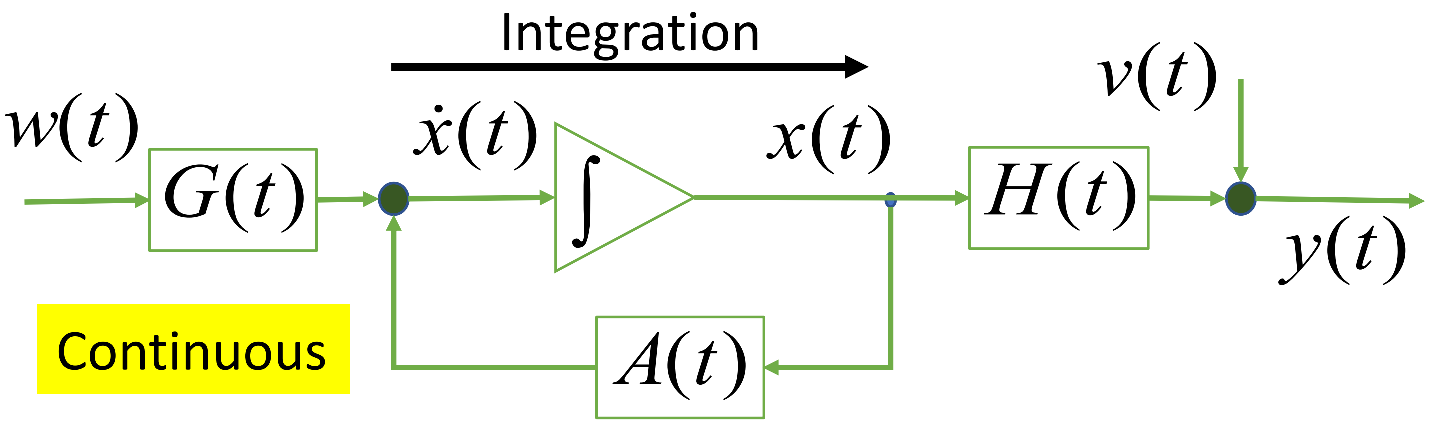
Modeling of noise

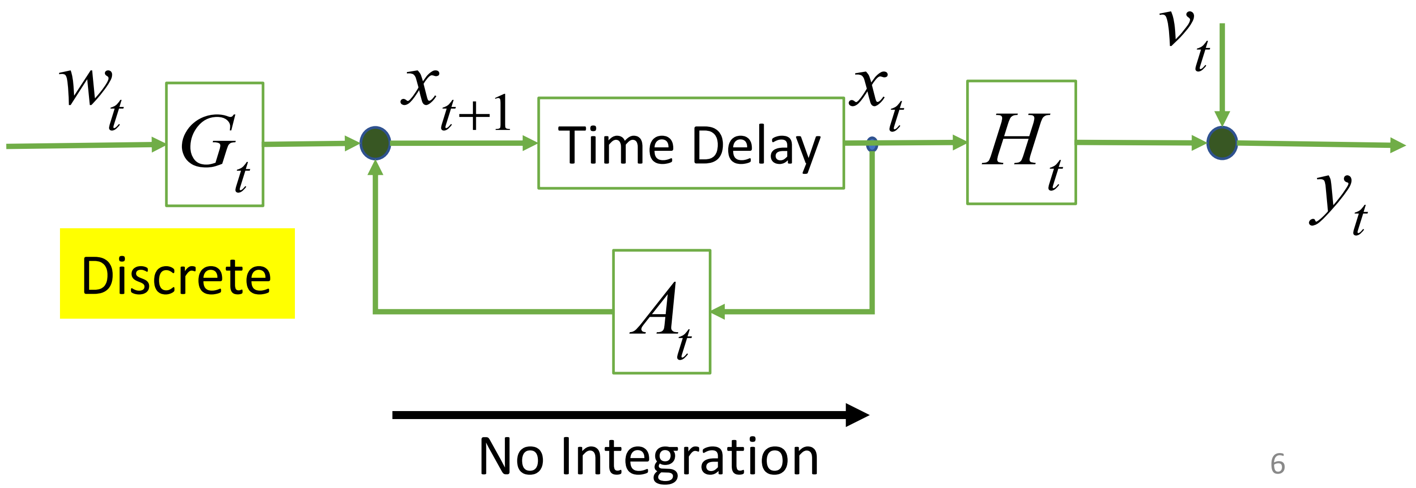
- Comparing the continuous time and discrete time systems, we find that the process noise going through the continuous time system is integrated, while the one through the discrete time system is not. Therefore,

$$w_t = \int_{t-\Delta t}^{t} w(\tau)d\tau = \overline{w}(t)\cdot \Delta t$$

- On the other hand, there is no such difference for measurement noise: v_t is merely the time average of $v(t)$.

$$v_t = \frac{1}{\Delta t}\int_{t-\Delta t}^{t} v(\tau)d\tau = \overline{v}(t)$$



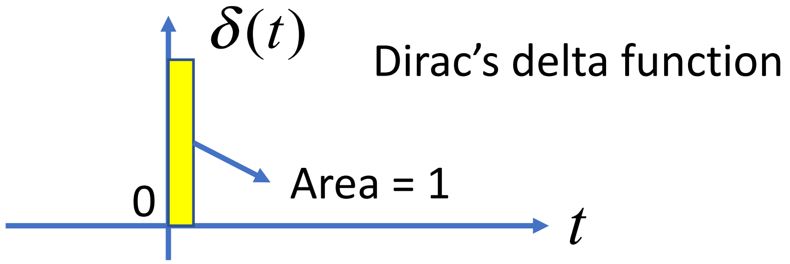


Noise Covariance in Discrete Time and Continuous Time Representation

- Measurement noise in discrete time v_t is the time average of continuous time noise $v(t)$ over sampling interval $\Delta\tau$.

$$v_t = \frac{1}{\Delta t}\int_{t-\Delta t}^{t} v(\tau)d\tau = \bar{v}(t)$$

- Based on this, the covariance of measurement noise is related to the one in continuous time as:



$$R_t = E[v_t v_t^T] = E\left[\int\int_{t-\Delta t}^{t} v(\tau)v^T(\tau')\,d\tau\,d\tau'\frac{1}{\Delta t^2}\right]$$

$$= \int\left\{\int_{t-\Delta t}^{t} \underbrace{E[v(\tau)v^T(\tau')]}_{R(\tau)\delta(\tau-\tau')}d\tau'\right\}d\tau\frac{1}{\Delta t^2} = \int_{t-\Delta t}^{t} R(\tau)\,d\tau\frac{1}{\Delta t^2} = \bar{R}(t)\Delta t\frac{1}{\Delta t^2} = \bar{R}(t)\frac{1}{\Delta t}$$

- Similarly, we can find the relationship between the process noise covariance in discrete time and the one in continuous time.

$$Q_t = E[w_t w_t^T] \cong \bar{Q}(t)\cdot\Delta t \qquad \text{where} \qquad \bar{Q}(t) = \frac{1}{\Delta t}\int_{t-\Delta t}^{t} Q(\tau)\,d\tau$$

Combining the Covariance Propagation Law with the Covariance Update Law

- The correspondence between discrete time terms and the continuous time terms can be summarized as follows.

$$A_t = I + F(t)\cdot \Delta t,\ H_t = H(t),\ R_t = \bar{R}(t)\frac{1}{\Delta t},\ Q_t = \bar{Q}(t)\cdot \Delta t,\ P_t = P(t)$$

- Therefore, the Kalman gain is expressed as:

$$K_t = P_t H_t^T R_t^{-1} = P_t H_t^T \bar{R}^{-1}(t)\cdot \Delta t = K(t)\Delta t \quad where \quad K(t) = P(t)H^T(t)R^{-1}(t)$$

- Covariance propagation and covariance update laws can be combined:

$$P_{t+1|t} = A_t P_t A_t^T + G_t Q_t G_t^T = A_t (I - K_t H_t) P_{t|t-1} A_t^T + G_t Q_t G_t^T$$

Combining the Covariance Propagation Law with the Covariance Update Law

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- Covariance propagation and covariance update laws can be combined:

$$P_{t+1|t} = A_t P_t A_t^T + G_t Q_t G_t^T = A_t(I - K_t H_t)P_{t|t-1}A_t^T + G_t Q_t G_t^T$$

- Using the above relationships, we can convert the combined covariance propagation and update law in discrete-time into:

$$\begin{aligned}P_{t+1|t} &= (I + F(t)\Delta t)(I - \Delta t\cdot K(t)H(t))P_{t|t-1}(I + F(t)\Delta t)^T + G(t)Q(t)\Delta t\, G^T(t)\\ &= P_{t|t-1} + \Delta t\cdot F(t)P_{t|t-1} - \Delta t\cdot K(t)H(t)P_{t|t-1} + P_{t|t-1}F^T(t)\Delta t\\ &\quad + G(t)Q(t)\Delta t\, G^T(t) + (higher\ order\ small\ quantities)\end{aligned}$$

$$P_{t+1|t} = P_{t|t-1} + \Delta t \cdot F(t) P_{t|t-1} - \Delta t \cdot K(t) H(t) P_{t|t-1} + P_{t|t-1} F^T(t) \Delta t$$
$$+ G(t) Q(t) \Delta t \, G^T(t) + (\textit{higher order small quantities})$$

❑ Moving $P_{t|t-1}$ to the left hand side and divide both sides by Δt

$$\frac{P_{t+1|t} - P_{t|t-1}}{\Delta t} = F(t) P_{t|t-1} + P_{t|t-1} F^T(t) - K(t) H(t) P_{t|t-1} + G(t) Q(t) G^T(t)$$

❑ As $\Delta t \to 0$, $\lim_{\Delta t \to 0} P_{t|t-1} = P_t = P(t)$

$$P_{t+1|t} = P_{t|t-1} + \Delta t \cdot F(t)P_{t|t-1} - \Delta t \cdot K(t)H(t)P_{t|t-1} + P_{t|t-1}F^T(t)\Delta t$$
$$+ G(t)Q(t)\Delta t\, G^T(t) + (\textit{higher order small quantities})$$

- Moving $P_{t|t-1}$ to the left hand side and divide both sides by Δt

$$\frac{P_{t+1|t} - P_{t|t-1}}{\Delta t} = F(t)P_{t|t-1} + P_{t|t-1}F^T(t) - K(t)H(t)P_{t|t-1} + G(t)Q(t)G^T(t)$$

- As $\Delta t \to 0$, $\lim_{\Delta t \to 0} P_{t|t-1} = P_t = P(t)$

$$\frac{dP(t)}{dt} = F(t)P(t) + P(t)F^T(t) - P(t)H^T(t)R^{-1}(t)H(t)P(t) + G(t)Q(t)G^T(t)$$

where we used $K(t) = P(t)H^T(t)R^{-1}(t)$

- This is called the **Riccati Differential Equation**. Note that this is a matrix equation. Since the covariance $P(t) \in \Re^{n\times n}$ is a symmetric matrix, $\frac{1}{2}n(n+1)$ independent scalar differential equations are involved.

Matrix Riccati Differential Equation

- Each term involved in the Matrix Riccati Differential Equation has a clear physical meaning.

$$\frac{dP(t)}{dt} = F(t)P(t) + P(t)F^T(t) - P(t)H^T(t)R^{-1}(t)H(t)P(t) + G(t)Q(t)G^T(t) \tag{62}$$

The effect of the unforced system dynamics upon the error covariance transition.

$$\frac{dx}{dt} = F(t)x(t) + G(t)w(t)$$

Matrix Riccati Differential Equation

- Each term involved in the Matrix Riccati Differential Equation has a clear physical meaning.

$$\frac{dP(t)}{dt} = \underbrace{F(t)P(t) + P(t)F^T(t)}_{\text{The effect of the unforced system dynamics upon the error covariance transition.}} - \underbrace{P(t)H^T(t)R^{-1}(t)H(t)P(t)}_{\text{Expected reduction of uncertainty as a result of state update using sensor signals having covariance } R(t).} + \underbrace{G(t)Q(t)G^T(t)}_{\text{Expected increase of uncertainty due to process noise with } Q(t).} \quad (62)$$

(These matrix products are positive semi-definite.)

The effect of the unforced system dynamics upon the error covariance transition.

$$\frac{dx}{dt} = F(t)x(t) + G(t)w(t)$$

Expected reduction of uncertainty as a result of state update using sensor signals having covariance $R(t)$.

Expected increase of uncertainty due to process noise with $Q(t)$.

- This Riccati equation is the key component determining the optimal state update gain, i.e. Kalman Gain: $K(t) = P(t)H^T(t)R^{-1}(t)$. It aggregates both state propagation and update laws, and represents how each of propagation and update contributes to the prediction uncertainty, together with the inherent dynamics of the process.

Kalman-Bucy Filter (Continuous-Time Kalman Filter) - 1961

Linear Time-Varying System

State Equation $$\frac{dx}{dt} = F(t)x(t) + G(t)w(t)$$

Measurement Equation $$y(t) = H(t)x(t) + v(t)$$

Assumed Observable.

Uncorrelated noise

$$E[v(t)v^T(s)] = \begin{cases} R(t); t = s \\ 0; \quad t \neq s \end{cases},$$

$$E[w(t)w^T(s)] = \begin{cases} Q(t); t = s \\ 0; \quad t \neq s \end{cases},$$

$$E[v(t)w^T(s)] = 0; \forall t, \forall s$$

State update & propagation $$\frac{d}{dt}\hat{x}(t) = F(t)\hat{x}(t) + K(t)[y(t) - \hat{y}(t)]$$

where $$K(t) = P(t)H^T(t)R^{-1}(t)$$ Kalman Gain

Covariance update & propagation

$$\frac{dP(t)}{dt} = F(t)P(t) + P(t)F^T(t) - P(t)H^T(t)R^{-1}(t)H(t)P(t) + G(t)Q(t)G^T(t)$$

The Riccati Differential Equation

6.2 Algebraic Riccati Equation (ARE)

- The Riccati differential equation is nonlinear. We aim to examine:
  - How does $P(t)$ evolve with time?
  - Does it converge?
  - If converging, will it converge to 0 or somewhere else? $P(t) \underset{t\to\infty}{\longrightarrow} 0$ or $P(\infty) \neq 0$

- Before analyzing the dynamic transition, we begin with steady-state properties.
- Assuming that the Matrix Riccati Differential Equation converges.

$$\frac{d}{dt}P(t) \underset{t\to\infty}{\longrightarrow} 0 \quad P(t) \underset{t\to\infty}{\longrightarrow} P_\infty, \quad F(t), G(t), H(t), Q(t), R(t) \underset{t\to\infty}{\longrightarrow} F, G, H, Q, R$$

- Under this assumption, the Riccati Differential Equation reduces to an algebraic equation.

$$0 = FP_\infty + P_\infty F^T - P_\infty H^T R^{-1} H P_\infty + GQG^T$$

- This is called the Algebraic Riccati Equation (ARE).

A Scalar Case of the Algebraic Riccati Equation

$$0 = FP_\infty + P_\infty F^T - P_\infty H^T R^{-1} H P_\infty + GQG^T$$

- Consider a scalar case where all the variables and parameters are scalar. The Algebraic Riccati Equation reduces to

$$2FP_\infty - \frac{H^2}{R} P_\infty^2 + G^2 Q = 0$$

- This is a simple 2nd order algebraic equation with the following solution.

$$P_\infty = \frac{R}{H^2}\left(F \pm \sqrt{F^2 + \frac{H^2}{R} G^2 Q} \right)$$

- By definition $P(t) \geq 0, P_\infty \geq 0$. Therefore, we discard the negative solution.

$$P_\infty = \frac{R}{H^2}\left(F + \sqrt{F^2 + \frac{H^2}{R} G^2 Q} \right)$$

A Scalar Case of the Algebraic Riccati Equation (Continued)

- Let's examine the solution: $P_\infty = \dfrac{R}{H^2}\left(F + \sqrt{F^2 + \dfrac{H^2}{R}G^2 Q}\right)$

Case 1. $F = 0$ $\quad A_t = I + F\Delta t = I$

This implies the estimation of constant parameter. $P_\infty = \dfrac{G}{H}\sqrt{RQ}$

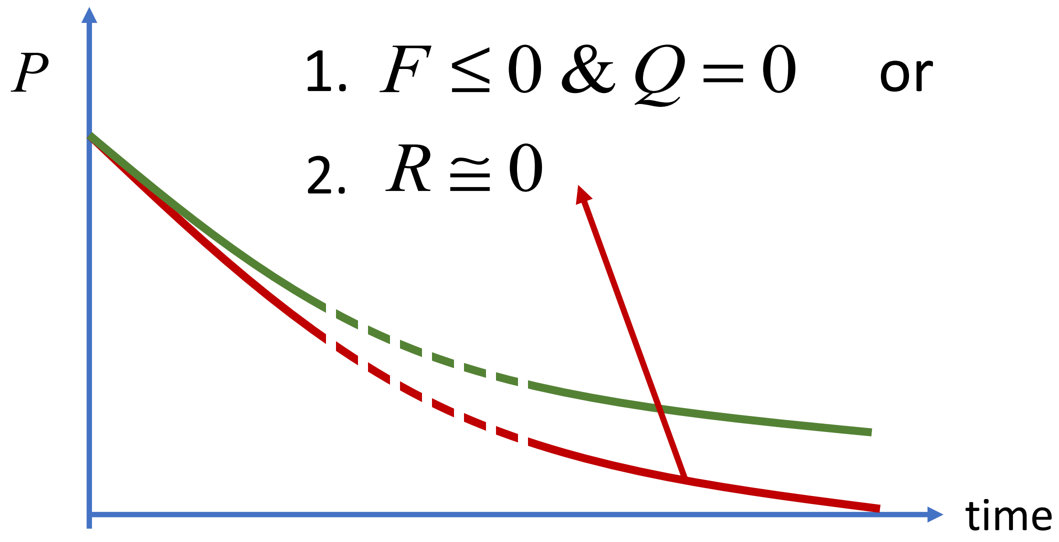
If R or Q is 0, then $P_\infty \cong 0$ assuming $GH > 0$.

Case 2. $Q = 0$ $\quad$ No process noise

$$P_\infty = \frac{R}{H^2}\left(F + \sqrt{F^2}\right)$$

2-a) F > 0, i.e. an unstable system $P_\infty = \dfrac{R}{H^2}2F$

2-b) F < 0, i.e. a stable system $P_\infty = 0$



Case 3. $R \cong 0$ $\quad$ A perfect sensor

$$P_\infty = \frac{R}{H^2}\left(F + \sqrt{F^2 + \frac{H^2}{R}G^2 Q}\right) \xrightarrow[R \to +0]{} 0$$

- Covariance never goes to zero, unless $F \le 0$ & $Q = 0$ or $R \cong 0$.

6.3 Convergence Analysis and Transient Response

- We have examined the steady-state solution to the Algebraic Riccati Equation, assuming that a solution exists. However, the original Matrix Riccati Differential Equation is nonlinear, simultaneous differential equations, the behaviors of which may be complex.
- Here, we examine the transient response of the differential equation and discuss conditions for obtaining a physically meaningful solution.
- We introduce a technique for solving the Matrix Riccati Differential Equation.

<u>Lemma</u> Matrix Fraction Decomposition

Suppose that the square matrix $P(t)$ in the Matrix Riccati Differential Equation is decomposed to

$$P(t) = A(t)B^{-1}(t), \quad \forall t$$

Where $A(t)$ and $B(t)$ are differentiable and $B(t)$ is non-singular. Then the Matrix Riccati Differential Equation, eq.(62), can be written in the following *linear* form.

$$\frac{d}{dt}\begin{pmatrix} A(t) \\ B(t) \end{pmatrix} = \begin{bmatrix} F(t) & G(t)Q(t)G^T(t) \\ H^T(t)R^{-1}(t)H(t) & -F^T(t) \end{bmatrix}\begin{pmatrix} A(t) \\ B(t) \end{pmatrix}$$

Proof of the Lemma

$$\frac{dP(t)}{dt} = F(t)P(t) + P(t)F^T(t) - P(t)H^T(t)R^{-1}(t)H(t)P(t) + G(t)Q(t)G^T(t)$$

Since matrix $B(t)$ is non-singular,

$$B(t)B^{-1}(t) = I \quad \rightarrow \quad \dot{B}B^{-1} + B\dot{B}^{-1} = 0 \quad \rightarrow \quad \dot{B}^{-1} = -B^{-1}\dot{B}B^{-1}$$

Differentiating $P(t) = A(t)B^{-1}(t)$ yields

$$\frac{dP(t)}{dt} = \dot{A}B^{-1} + A\dot{B}^{-1} = \dot{A}B^{-1} - AB^{-1}\dot{B}B^{-1} \tag{74}$$

Substituting $P(t) = A(t)B^{-1}(t)$ into the Riccati Equation yields.

$$\frac{dP(t)}{dt} = FAB^{-1} + AB^{-1}F^T - AB^{-1}H^T R^{-1}HAB^{-1} + GQG^T \tag{75}$$

Comparing (74) and (75), we find

$$\dot{A}B^{-1} - AB^{-1}\dot{B}B^{-1} = FAB^{-1} + AB^{-1}F^T - AB^{-1}H^T R^{-1}HAB^{-1} + GQG^T$$

Post-multiplying B to both sides,

$$\dot{A} - AB^{-1}\dot{B} = FA + AB^{-1}F^T B - AB^{-1}H^T R^{-1}HA + GQG^T B$$

$$\dot{A} - AB^{-1}\dot{B} = FA + AB^{-1}F^T B - AB^{-1}H^T R^{-1}HA + GQG^T B$$

Collecting terms, we obtain:

$$\frac{dA}{dt} - AB^{-1}\frac{dB}{dt} = (FA + GQG^T B) - AB^{-1}(H^T R^{-1}HA - F^T B)$$

Comparing corresponding terms on both sides, consider the following two differential equations:

$$\frac{dA}{dt} = (FA + GQG^T B), \quad \frac{dB}{dt} = H^T R^{-1}HA - F^T B)$$

These two matrix differential equations can be combined as

$$\frac{d}{dt}\begin{pmatrix} A(t) \\ B(t) \end{pmatrix} = \begin{bmatrix} F(t) & G(t)Q(t)G^T(t) \\ H^T(t)R^{-1}(t)H(t) & -F^T(t) \end{bmatrix}\begin{pmatrix} A(t) \\ B(t) \end{pmatrix}$$

The matrix of the above linear differential equation is a Hamiltonian Matrix. □

Punchline If we find $A(t)$ and $B(t)$ that satisfy the above linear differential equation, then $P(t) = A(t)B^{-1}(t)$ is a solution to the Riccati Differential Equation.

- Note that the above differential equation is linear, although the original Riccati equation is nonlinear.
- Using this linear equation, we can investigate properties of the Riccati equation.
- Let us first consider a scalar case: $P(t) = a(t) / b(t)$

  where $a(\mathrm{t})$ and $b(\mathrm{t})$ are scalar functions and $b(t) \neq 0$. We also assume that the system is time-invariant with all parameters being constant: $F, H, G, Q, and\ R.$

$$\begin{pmatrix} \dot{a} \\ \dot{b} \end{pmatrix} = \begin{pmatrix} F & G^2 Q \\ \dfrac{H^2}{R} & -F \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

  - This linear differential equation with constant parameters can be solved without difficulty. First computing the eigenvalues of the Hamiltonian matrix yield.

$$\lambda_{1,}\lambda_2 = \pm\sqrt{F^2 + \frac{Q}{R} G^2 H^2} = \pm\lambda$$

- The solution is given by

$$\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = e^{\mathbf{M}\cdot t}\begin{pmatrix} P_0 \\ 1 \end{pmatrix}$$

where initial conditions are

$$a(0) = P_0 \quad \text{and} \quad b(0) = 1$$

and $\boldsymbol{M}$ is the Hamiltonian Matrix above, which can be diagonalized using eigen vectors associated with the eigenvalues.

$$\mathbf{M} = [\mathbf{v}_1, \mathbf{v}_2]\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}[\mathbf{v}_1, \mathbf{v}_2]^{-1}$$

$$\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = [\mathbf{v}_1, \mathbf{v}_2]\cdot\begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}[\mathbf{v}_1, \mathbf{v}_2]^{-1}\begin{pmatrix} P_0 \\ 1 \end{pmatrix}$$

- This leads to

$$a(t) = \frac{1}{2\lambda}\left\{[P_0(\lambda+F)+q]e^{\lambda t} + [P_0(\lambda-F)-q]e^{-\lambda t}\right\}$$

$$b(t) = \frac{1}{2\lambda q}\left\{(\lambda-F)[P_0(\lambda+F)+q]e^{\lambda t} - (\lambda+F)[P_0(\lambda-F)-q]e^{-\lambda t}\right\}$$

where $q = G^2Q$. Therefore, the covariance is given by

$$P(t) = \frac{a(t)}{b(t)} = q\frac{[P_0(\lambda+F)+q]+[P_0(\lambda-F)-q]e^{-2\lambda t}}{(\lambda-F)[P_0(\lambda+F)+q]-(\lambda+F)[P_0(\lambda-F)-q]e^{-2\lambda t}}$$

- The steady-state solution is given by

$$P_\infty = \lim_{t\to\infty} P(t) = \frac{q}{\lambda-F} = \frac{R}{H^2}\left(F + \sqrt{F^2 + \frac{Q}{R}H^2G^2}\right)$$

This agrees with the solution to the Algebraic Riccati Equation.

Numerical Example

$$P(t)=\frac{a(t)}{b(t)}=q\frac{[P_0(\lambda+F)+q]+[P_0(\lambda-F)-q]e^{-2\lambda t}}{(\lambda-F)[P_0(\lambda+F)+q]-(\lambda+F)[P_0(\lambda-F)-q]e^{-2\lambda t}}$$

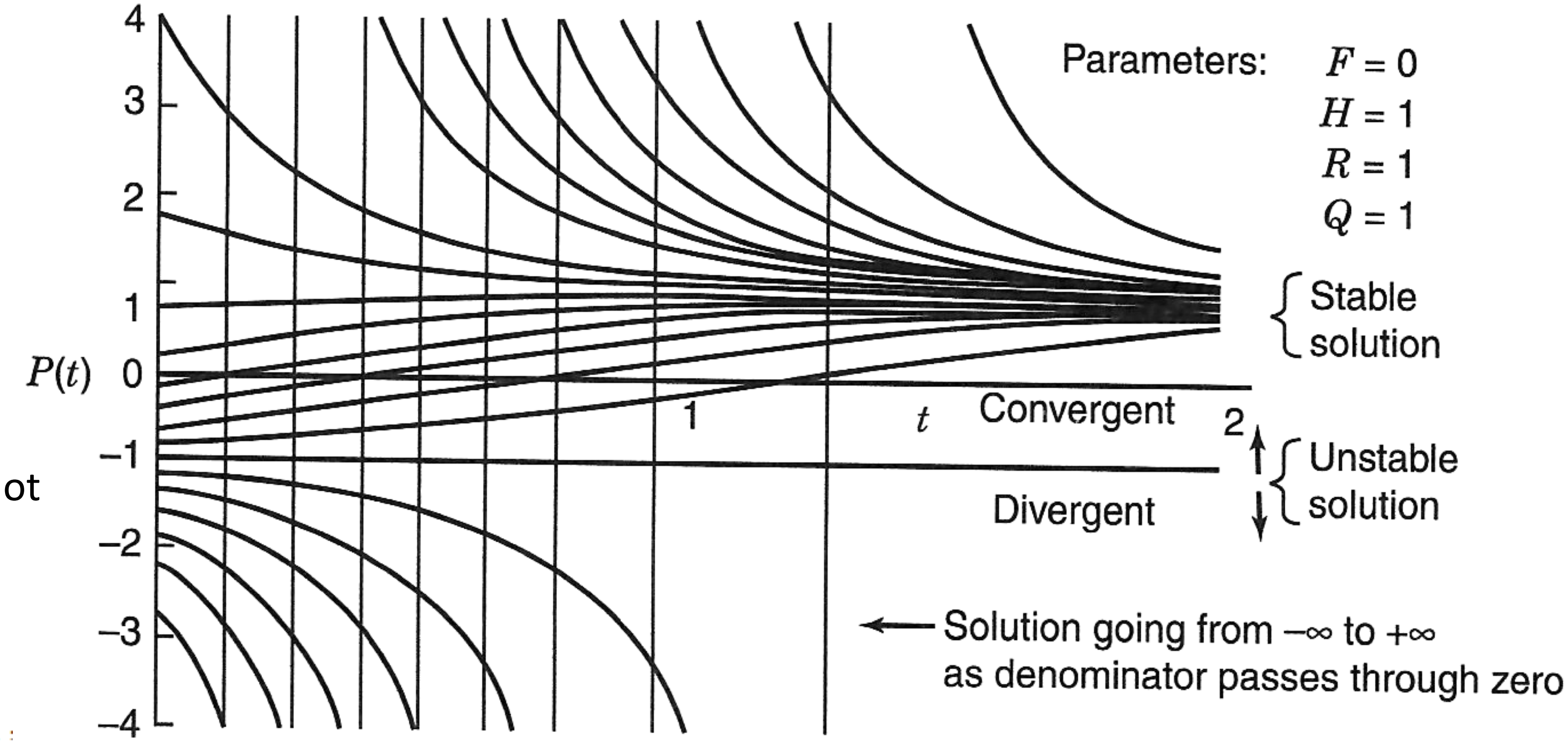
- Note that the denominator may become zero at time:

$$t_{diverge}=\frac{1}{2\lambda}\ln\frac{(\lambda+F)[P_0(\lambda-F)-q]}{(\lambda-F)[P_0(\lambda+F)+q]}$$

- This implies that the solution is discontinuous, going from negative infinite to positive infinite when passing the zero point.
- This undesirable discontinuity does not occur when it starts with an initial condition

$$P_0>\frac{R}{H^2}\left(F-\sqrt{F^2+\frac{H^2}{R}G^2Q}\right)\triangleq P_-$$

Larger than the negative solution of ARE.



From Grewal and Andrews, "Kalman Filtering", Chapter 4.8, Wiley 2001.

An important property of the Riccati Differential Equation (RDE):

If the system is observable, i.e. (F, H), Observable Pair, then the RDE has a positive-definite, symmetric solution for an arbitrary positive-definite initial value of matrix P_o>0;

$$\exists P(t) \; for \; \forall P_0 > 0 \; p.d., \quad \text{such that } P(t) > 0 \, p.d., P(t) = P^T(t) \in R^{n \times n}, \quad \forall t > 0, \tag{87}$$

Example 1

Consider a stochastic system in continuous time modeled by the equations:

$$\dot{x}(t) = -x(t) + w(t)$$

$$y(t) = x(t) + v(t)$$

where both process noise $w(t)$ and measurement noise $v(t)$ are white Gaussian with

$$w(t) \sim N(0, 30)$$

$$v(t) \sim N(0, 20)$$

a). Obtain the Riccati differential equation associated with the continuous-time Kalman filter for this system, and solve it for the steady-state value of $P(t)$, given initial condition P_o.

b). Solve the Riccati differential equation using the fraction decomposition method discussed in class: $P(t) = a(t)/b(t)$. Use initial conditions of $a(0) = P_0$, $b(0) = 1$.

Solutions

a) The Riccati differential equation $\frac{d}{dt}P(t) = FP(t) + P(t)F^T - P(t)H^T R^{-1} HP(t) + GQG^T$

For this system *F=-1, H=1, R=20*, and *Q=30*

$$\dot{P}(t) = -2P(t) - \frac{1}{20}P^2(t) + 30$$

Steady state $0 = FP_\infty + P_\infty F^T - P_\infty H^T R^{-1} HP_\infty + GQG^T$

Steady state solution

$$P_\infty = \frac{R}{H^2}\left(F + \sqrt{F^2 + \frac{Q}{R}H^2G^2}\right)$$

$$= \frac{20}{1}\left(-1 + \sqrt{1 + \frac{30}{20}}\right) = -20 + 10\sqrt{10}$$

b) $P(t)=a(t)/b(t)$ if we find a and b that satisfy:

$$\begin{pmatrix}\dot a\\ \dot b\end{pmatrix}=\begin{bmatrix}F & G^2Q\\ H^2/R & -F\end{bmatrix}\begin{pmatrix}a\\ b\end{pmatrix}=\begin{bmatrix}-1 & 30\\ 1/20 & 1\end{bmatrix}\begin{pmatrix}a\\ b\end{pmatrix}$$

then $P(t)=a(t)/b(t)$ satisfies the Riccati differential equation.

The eigenvalues of the Hamiltonian Matrix are

$$\lambda_{1,}\lambda_2=\pm\sqrt{F^2+\frac{Q}{R}G^2H^2}=\pm\frac{\sqrt{10}}{2}=\pm\lambda$$

$$P(t)=\frac{Q\left[P_0(\lambda+F)+Q\right]+Q\left[P_0(\lambda-F)-G^2Q\right]e^{-2\lambda t}}{(\lambda-F)\left[P_0(\lambda+F)+Q\right]-(\lambda+F)\left[P_0(\lambda-F)-G^2Q\right]e^{-2\lambda t}}$$

$$\lim_{t\to\infty}P(t)=P_\infty=\frac{Q}{\lambda-F}=\frac{30}{\frac{\sqrt{10}}{2}+1}=-20+10\sqrt{10}$$

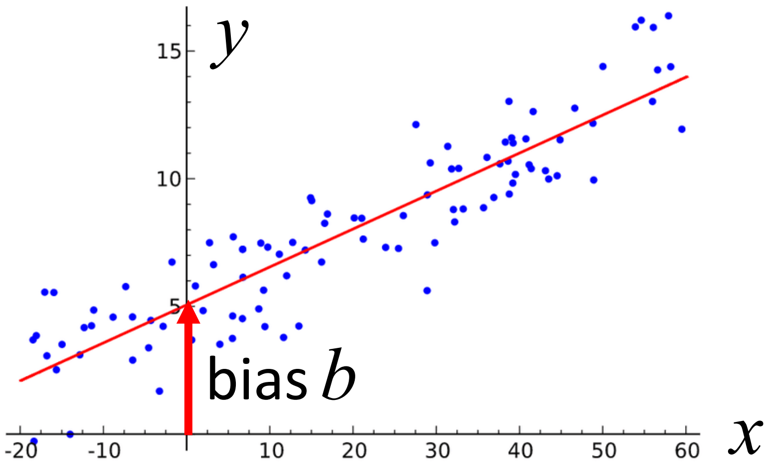
Example 2 Consider a scalar stochastic system in continuous time given by:

$$\dot{x}(t) = ax(t) + w(t)$$
$$y(t) = cx(t) + v(t)$$

where a and c are known constants, and $w(t)$ and $v(t)$ are, respectively, uncorrelated process noise and measurement noise with zero mean values and variances Q and R. The sensor used for measuring output $y(t)$ tends to have bias b, which is assumed to be constant but is unknown. We want to build a Kalman filter to estimate both state $x(t)$ and unknown bias b simultaneously by using an augmented state vector:

$$X = \begin{pmatrix} x \\ b \end{pmatrix}$$

Answer the following questions.



Questions

a). Obtain revised state equation and measurement equation associated with the augmented state $X(t)$.

b). Obtain the Riccati differential equation associated with this Kalman filter, and solve it for the steady-state value of $P(t)$, given initial condition P_o.

c). Discuss whether the bias can be estimated correctly, and under which conditions the state estimation error converges to 0. Obtain properties of the system and parameter values that allow $P(t)$ to converge to 0: $\lim_{t\to\infty} P(t) = 0$.

$$X = \begin{pmatrix} x \\ b \end{pmatrix}$$

state

parameter

There is no fundamental difference between state and parameter.

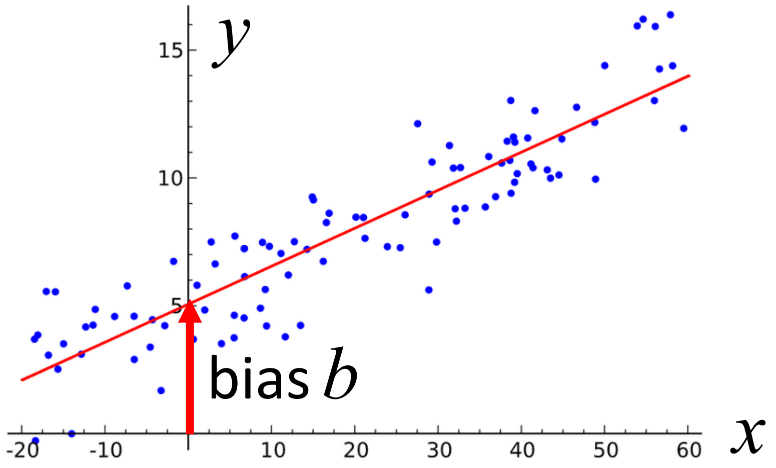
Solution

a). The bias is on the sensor. Therefore, the output function alone must be modified. The state equation remains the same except that the dynamics of the new state variable, i.e. the bias b, must be added:

$$\dot{x}(t) = ax(t) + w(t)$$

$$\dot{b}(t) = 0$$

$$\therefore \quad \frac{d}{dt}\begin{pmatrix} x(t) \\ b(t) \end{pmatrix} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} x(t) \\ b(t) \end{pmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w(t)$$



The observed output using this sensor is given by:

$$y(t) = cx(t) + b(t) + v(t)$$

$$\therefore \quad y(t) = \begin{bmatrix} c & 1 \end{bmatrix}\begin{pmatrix} x(t) \\ b(t) \end{pmatrix} + v(t)$$

This gives the output equation in the augmented state space.

Question b). Obtain the Riccati differential equation associated with this Kalman filter, and solve it for the steady-state value of $P(t)$, given initial condition P_o.

Solution

$$F=\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix},\quad G=\begin{bmatrix} 1 \\ 0 \end{bmatrix},\quad H=\begin{bmatrix} c & 1 \end{bmatrix}$$

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ b(t) \end{pmatrix}=\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} x(t) \\ b(t) \end{pmatrix}+\begin{bmatrix} 1 \\ 0 \end{bmatrix}w(t)$$

$$y(t)=\begin{bmatrix} c & 1 \end{bmatrix}\begin{pmatrix} x(t) \\ b(t) \end{pmatrix}+v(t)$$

Writing the covariance matrix as

$$P=\begin{pmatrix} p_1 & p_3 \\ p_3 & p_2 \end{pmatrix},$$

2 x 2 matrix

we can compute each term involved in the Riccati Differential Equation:

$$FP=\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} p_1 & p_3 \\ p_3 & p_2 \end{pmatrix}=\begin{bmatrix} ap_1 & ap_3 \\ 0 & 0 \end{bmatrix},\quad PF^T=\begin{bmatrix} ap_1 & 0 \\ ap_3 & 0 \end{bmatrix}$$

$$PH^T=\begin{pmatrix} p_1 & p_3 \\ p_3 & p_2 \end{pmatrix}\begin{bmatrix} c \\ 1 \end{bmatrix}=\begin{bmatrix} cp_1+p_3 \\ cp_3+p_2 \end{bmatrix}$$

$$PH^TR^{-1}HP=\frac{1}{R}\begin{bmatrix} cp_1+p_3 \\ cp_3+p_2 \end{bmatrix}\begin{bmatrix} cp_1+p_3 & cp_3+p_2 \end{bmatrix}$$

$$\frac{dP(t)}{dt}=F(t)P(t)+P(t)F^T(t)-P(t)H^T(t)R^{-1}(t)H(t)P(t)+G(t)Q(t)G^T(t)$$

Substituting these into the Riccati Differential equation yields:

$$\frac{d}{dt}\begin{pmatrix} p_1 & p_3 \\ p_3 & p_2 \end{pmatrix}=\begin{bmatrix} 2ap_1 & ap_3 \\ ap_3 & 0 \end{bmatrix}-\frac{1}{R}\begin{bmatrix} (cp_1+p_3)^2 & (cp_1+p_3)(cp_3+p_2) \\ (cp_1+p_3)(cp_3+p_2) & (cp_3+p_2)^2 \end{bmatrix}+\begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix}$$

$$\frac{d}{dt}\begin{pmatrix} p_1 & p_3 \\ p_3 & p_2 \end{pmatrix} = \begin{bmatrix} 2ap_1 & ap_3 \\ ap_3 & 0 \end{bmatrix} - \frac{1}{R}\begin{bmatrix} (cp_1+p_3)^2 & (cp_1+p_3)(cp_3+p_2) \\ (cp_1+p_3)(cp_3+p_2) & (cp_3+p_2)^2 \end{bmatrix} + \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix}$$

At steady state, $\frac{dP}{dt} = 0$. Three simultaneous equations are obtained from the Algebraic Riccati equation:

$$0 = 2ap_1 - \frac{1}{R}(cp_1+p_3)^2 + Q \qquad (1)$$

$$0 = ap_3 - \frac{1}{R}(cp_1+p_3)(cp_3+p_2) \qquad (2)$$

$$0 = -\frac{1}{R}(cp_3+p_2)^2 \qquad (3)$$

From (3), $cp_3 + p_2 = 0$. Substituting this into (2) yields $p_3 = 0$. Therefore, $p_2 = -cp_3 = 0$.

$$\frac{c^2}{R}p_1^2 - 2ap_1 - Q = 0$$

Solving this and taking the positive solution yield

$$p_1 = \frac{R}{c^2}\left(a + \sqrt{a^2 + \frac{Q}{R}c^2}\right)$$

$$P_\infty = \begin{bmatrix} \frac{R}{c^2}\left(a + \sqrt{a^2 + \frac{Q}{R}c^2}\right) & 0 \\ 0 & 0 \end{bmatrix}$$

Question c). Discuss whether the bias can be estimated correctly, and under which conditions the state estimation error converges to 0. Obtain properties of the system and parameter values that allow $P(t)$ to converge to 0: $\lim_{t\to\infty} P(t) = 0$.

Solution

- Since $p_3 = 0$ at steady state, the estimation of the sensor bias b converges to the correct value; zero estimation error, regardless of other parameters. Note that this is possible because no process noise is involved in the bias dynamics.
- The estimation error of state $x(t)$ becomes zero, $p_1 = 0$, when either of the following two conditions is met:
    - Zero sensor noise: $R = 0$.
    - No process noise: Q = 0, and the system is stable or marginally stable: $a \le 0$.

$$P_\infty = \begin{bmatrix} \frac{R}{c^2}\left(a + \sqrt{a^2 + \frac{Q}{R}c^2}\right) & 0 \\ 0 & 0 \end{bmatrix} = \begin{pmatrix} p_1 & p_2 \\ p_2 & p_3 \end{pmatrix}$$

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ b(t) \end{pmatrix} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} x(t) \\ b(t) \end{pmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w(t)$$

$$y(t) = \begin{bmatrix} c & 1 \end{bmatrix}\begin{pmatrix} x(t) \\ b(t) \end{pmatrix} + v(t)$$