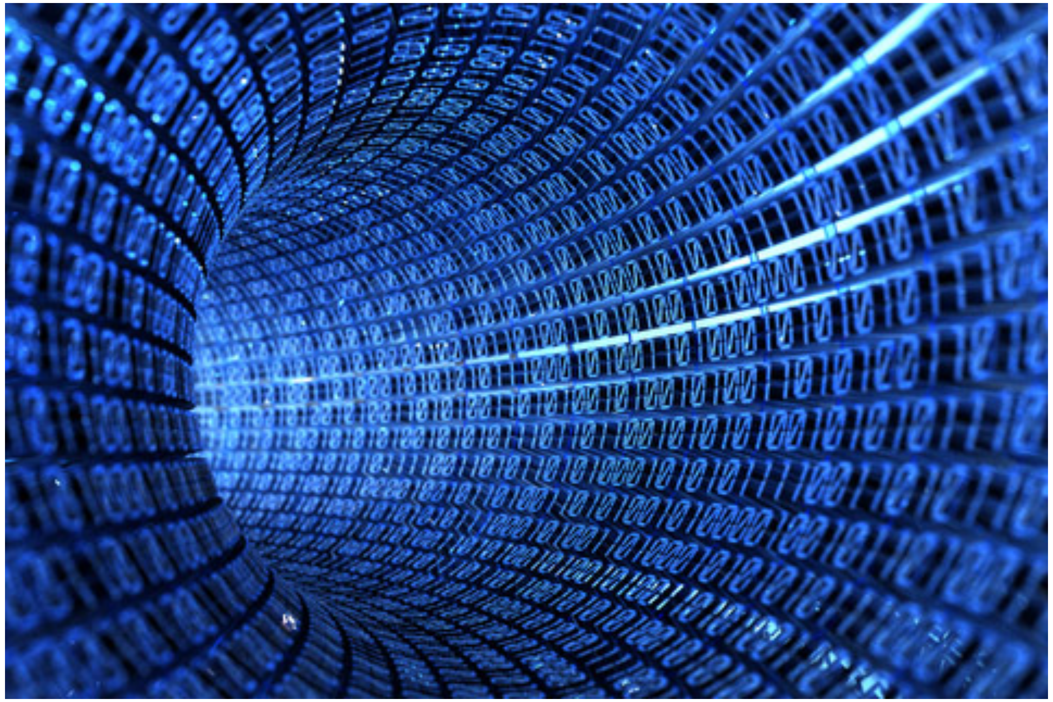
2.160 Identification, Estimation, and Learning

Part 3 Linear System Identification

Lecture 16

Unbiased Identification:

FIR Model Identification Using Laguerre Series Expansion



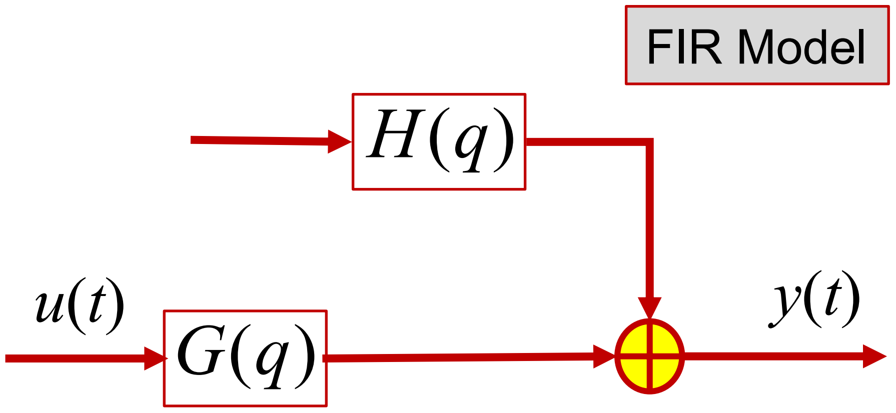
H. Harry Asada
Department of Mechanical Engineering
MIT

Consistent Estimate / Unbiased Estimate

$$\hat{\theta}_N \underset{N\to\infty}{\to} \theta_0 \qquad\qquad E[\hat{\theta}_N] = \theta_0$$

- Consider a Finite Impulse Response (FIR) Model with colored noise.

$$y(t) = b_1 u(t-1) + b_2 u(t-2) + \cdots + b_{n_b} u(t-n_b) + v(t)$$



Finite Impulse Response model

- Suppose that the goal is to identify G(q) only; no need to identify the noise dynamics: v(t) = H(q) e(t).
- Given a data set: $\{(u(t), y(t)) \mid t = 1, \cdots, N\}$
- The Least Squares Estimate is given by

$$\hat{\theta}_N = \arg\min_{\theta} \frac{1}{2N} \sum_{t=1}^{N} (y(t) - \varphi(t)^T \theta)^2 = R^{-1} \left(\frac{1}{N} \sum_{t=1}^{N} y(t)\varphi(t) \right), \quad R = \frac{1}{N} \sum_{t=1}^{N} \varphi(t)\varphi(t)^T$$

- Let θ_0 be the true parameter values; the data are generated by $y(t) = \varphi(t)^T \theta_0 + v(t)$
- Substituting this into the above LSE yields

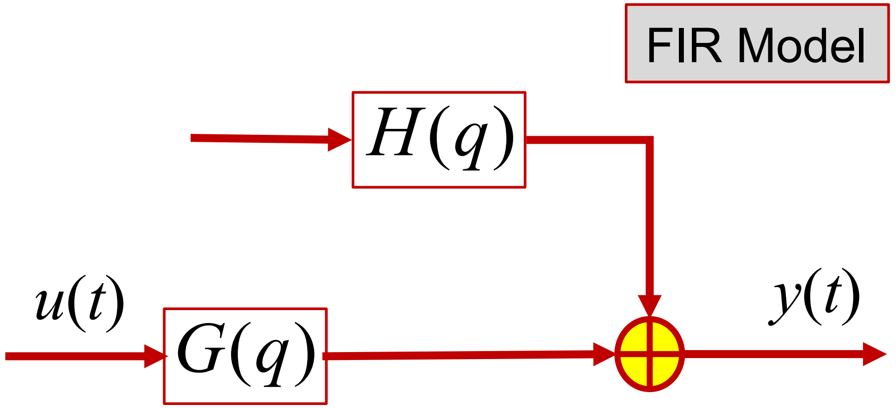
$$\hat{\theta}_N = R^{-1} \left(\frac{1}{N} \sum_{t=1}^{N} (\varphi(t)^T \theta_0 + v(t))\varphi(t) \right) = R^{-1} \underbrace{\frac{1}{N} \sum_{t=1}^{N} (\varphi(t)\varphi(t)^T)}_{R} \theta_0 + R^{-1} \frac{1}{N} \sum_{t=1}^{N} v(t)\varphi(t) = \theta_0 + R^{-1} \frac{1}{N} \sum_{t=1}^{N} v(t)\varphi(t)$$

Consistent Estimate / Unbiased Estimate

- Least Squares Estimate

$$\hat{\theta}_N = \theta_0 + R^{-1}\frac{1}{N}\sum_{t=1}^{N} v(t)\varphi(t)$$

- If the model is ARX, the regressor φ(t) includes y(t-1), y(t-2)… Therefore, <u>colored</u> noise v(t) may be correlated with the regressor, leading to Biased Estimate.



- On the other hand, if the model is FIR:

$$\varphi(t) = (u(t-1), u(t-2), \cdots, u(t-n_b))^T$$

$$y(t) = b_1 u(t-1) + b_2 u(t-2) + \cdots + b_{n_b} u(t-n_b) + v(t)$$

$$y(t) = \varphi(t)^T \theta_0 + v(t)$$

- The FIR's regressor φ(t) does not include y(t-1), y(t-2),….

- As long as u(t-i) is uncorrelated with noise v(t),

$$\frac{1}{N}\sum_{t=1}^{N} v(t)\varphi(t) = 0$$

- Therefore, Unbiased estimate is guaranteed. $\hat{\theta}_N \underset{N\to\infty}{\rightarrow} \theta_0 \quad E[\hat{\theta}_N] = \theta_0$

Pros and Cons of FIR System Identification

Pros

- ❑ FIR and Least Squares Estimate provide Unbiased/Consistent Estimate, although noise v(t) is colored.

$$\hat{\theta}_N \underset{N \to \infty}{\longrightarrow} \theta_0$$

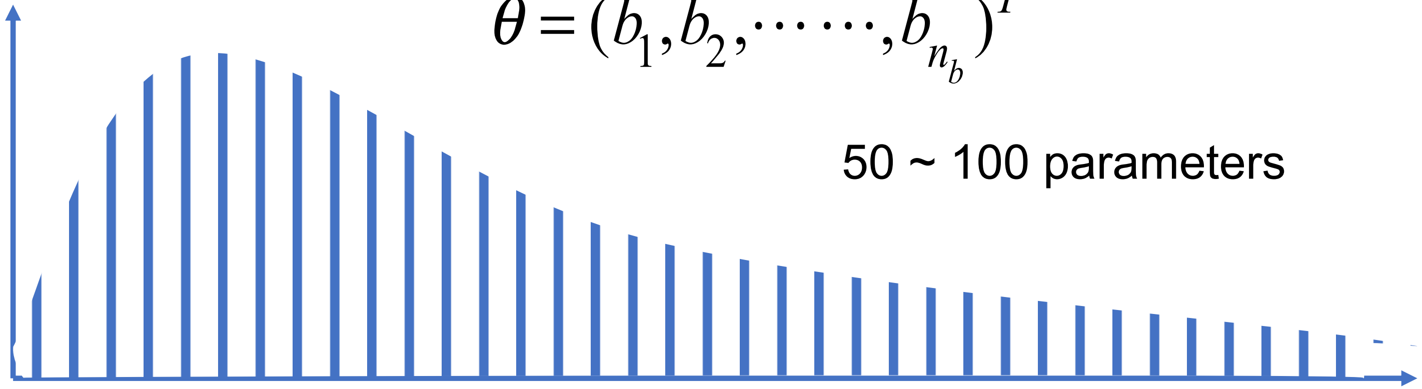
Cons

- ❑ FIR tends to have many parameters to identify: $n_b \gg 1$

Slowly-decaying mode

$$\theta = (b_1, b_2, \cdots\cdots, b_{n_b})^T$$

Impulse Response



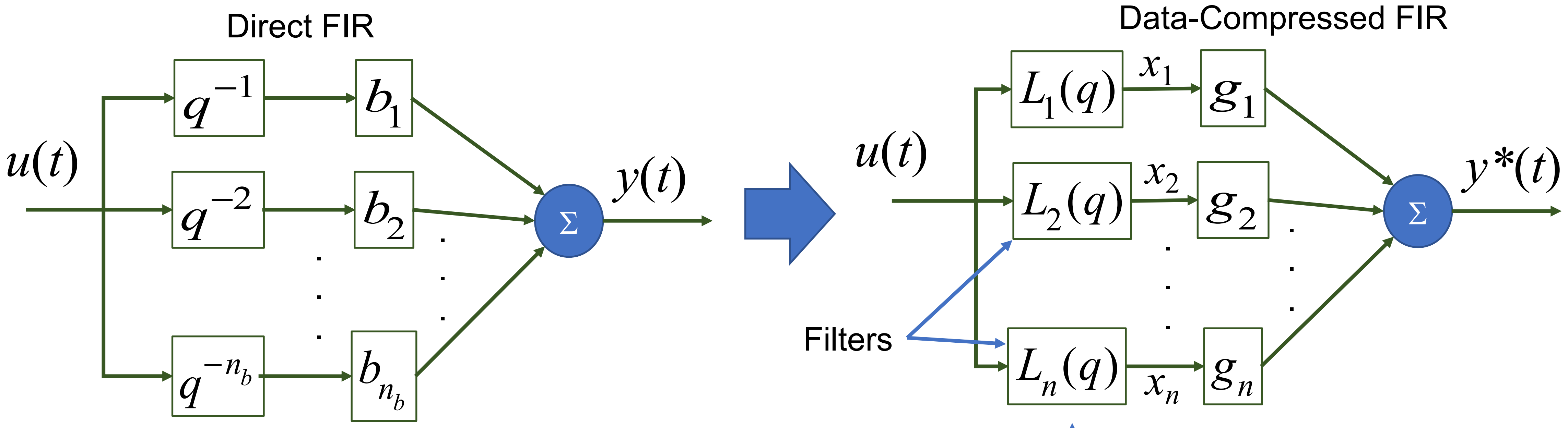
50 ~ 100 parameters

Time

- ❑ Slow convergence
- ❑ Difficult to meet Persistently Exciting conditions

Solution

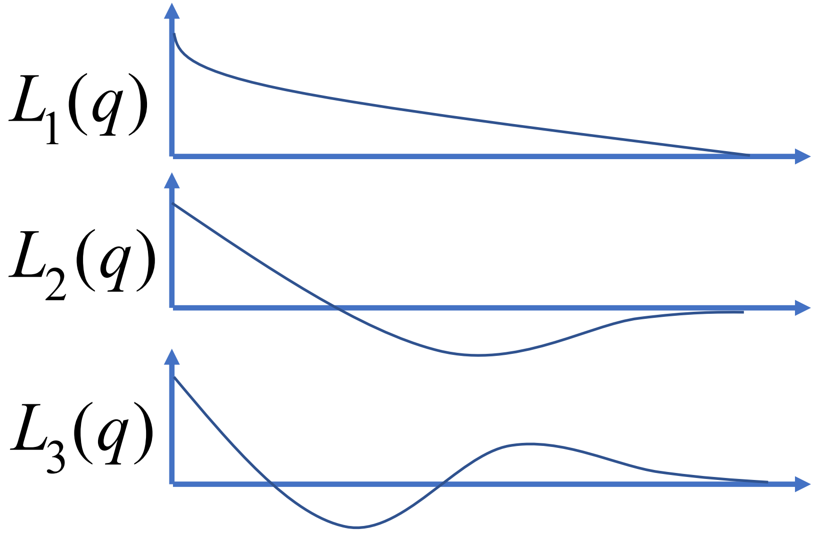
- Time-series data compression is an effective technique for solving this problem.



- Input signals are filtered with a series of special filters such that the order of FIR may be reduced:

$$y^*(t) = \sum_{k=1}^{n} g_k x_k(t), \quad n \ll n_b,$$

where $\qquad x_k(t) = L_k(q)\, u(t)$



Review of Z-Transform and Complex Functions

(Lecture Notes No.10, Section 10.2.2)

- Transfer function using time-shift operator, q.

$$G(q) = \sum_{k=0}^{\infty} g(k) q^{-k}$$

Time delay → $e^{\Delta t \cdot s}$ Δt = sampling interval

$$y(t) = G(q)u(t) \xrightarrow{\text{Laplace Transform}} L[y] = \sum_{k=0}^{\infty} g(k) e^{-\Delta t \cdot sk} \cdot L[u]$$

- Replacing $e^{\Delta t \cdot s}$ by z, we have the z-transform of the transfer function:

$$G(z) = \sum_{k=0}^{\infty} g(k) z^{-k}$$

…….. a complex function of $z = e^{\Delta t \cdot s}$

- Poles and Zeros

Zero: a complex number z_i that makes transfer function G(z) zero: $G(z_i) = 0$

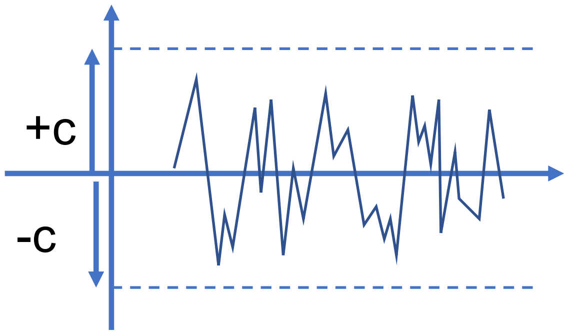
Pole: a complex number z_j that makes transfer function G(z) infinite: $G(z_j) = \infty$

Zero

$$G(s) = \frac{b_1 s + b_2}{s^2 + a_1 s + a_2}$$

Poles

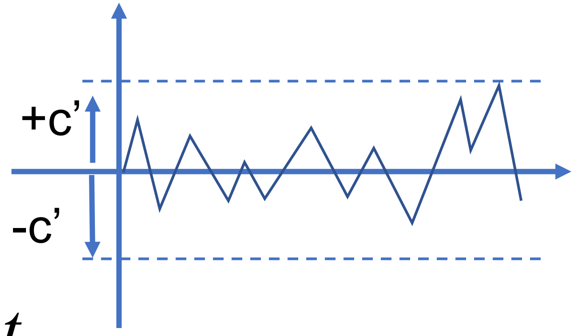
Bounded-Input, Bounded-Output Stability (BIBO)



$u(t)$ → Linea Time-Invariant System → $y(t)$

$|u(t)| \le c \quad \forall t$

$|y(t)| \le c' \quad \forall t$



Theorem

Transfer function $G(q) = \sum_{k=0}^{\infty} g(k) q^{-k}$ is BIBO stable, if $\sum_{k=0}^{\infty} |g(k)| < \infty$

Proof

$$|y(t)| = \left| \sum_{k=0}^{\infty} g(k) u(t-k) \right| \le \sum_{k=0}^{\infty} \left| g(k) u(t-k) \right| = \sum_{k=0}^{\infty} \left| g(k) \right| \cdot \left| u(t-k) \right| \le \sum_{k=0}^{\infty} \left| g(k) \right| \cdot c \le c'$$

Therefore, for any bounded input sequence, the output is bounded.

$|u(t)| \le c \quad \forall t$

Poles of BIBO-stable systems

- Associated with $G(q)$, consider

$$G(z) = \sum_{k=0}^{\infty} g(k) z^{-k}$$

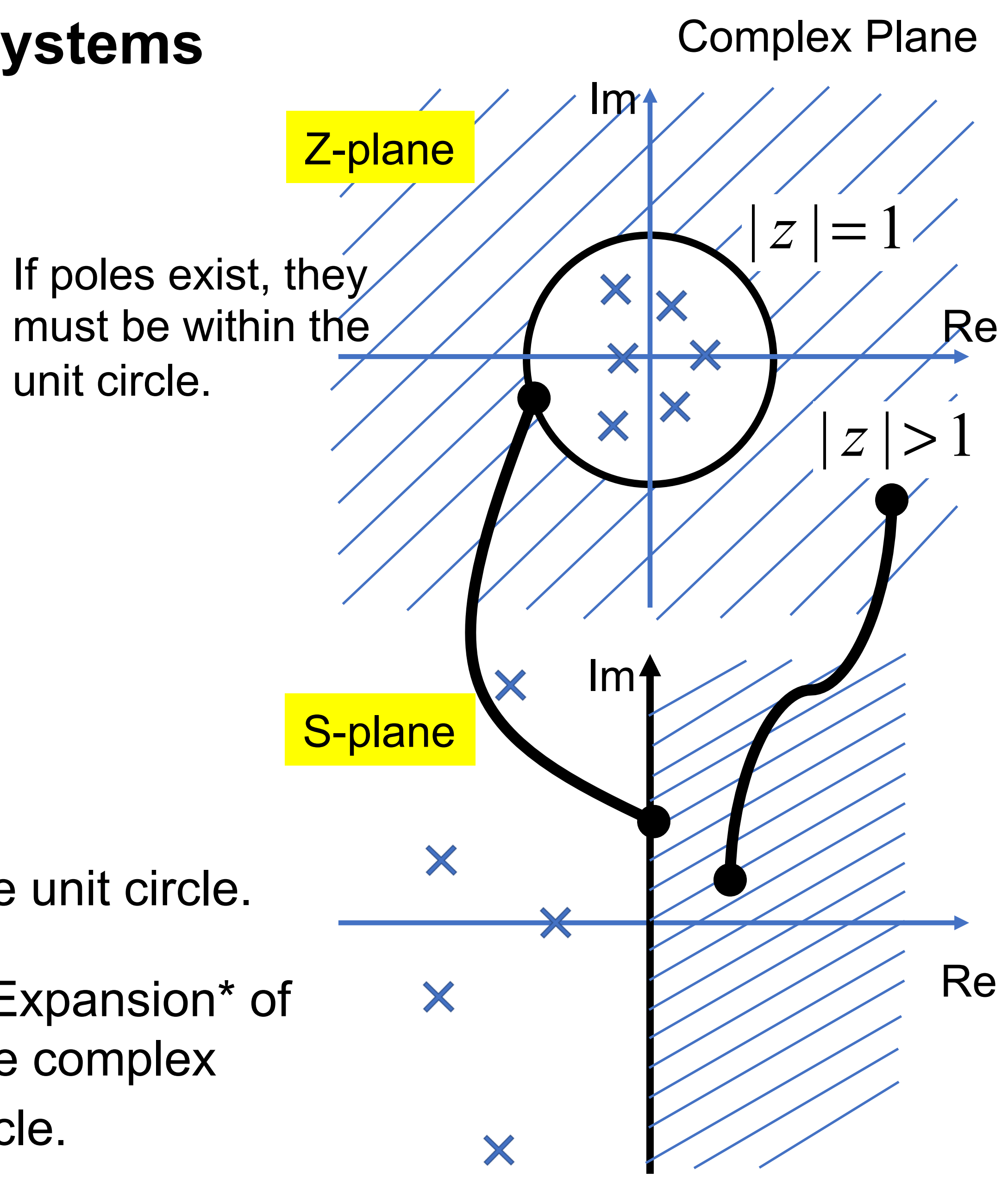
Then $$|G(z)| \le \sum_{k=0}^{\infty} |g(k)| \cdot |z|^{-k}$$

- If $G(q)$ is BIBO stable, for $|z|^{-1} \le 1 \quad (|z| \ge 1)$

$$|G(z)| \le \sum_{k=0}^{\infty} |g(k)| < \infty$$

- This implies that there is no pole on and outside the unit circle.

- Treating $G(z) = \sum g(k) z^{-k}$ as the Laurent Series Expansion* of a complex function, the above results mean that the complex function $G(z)$ is <u>analytic</u> on and outside the unit circle.



* This function includes terms of negative degree to which Taylor series expansion cannot be applied.

15.2 Continuous Time Laguerre Series Expansion

(Bilinear Transformation in Signal Processing)

Theorem 1. If a transfer function $G(s)$ is

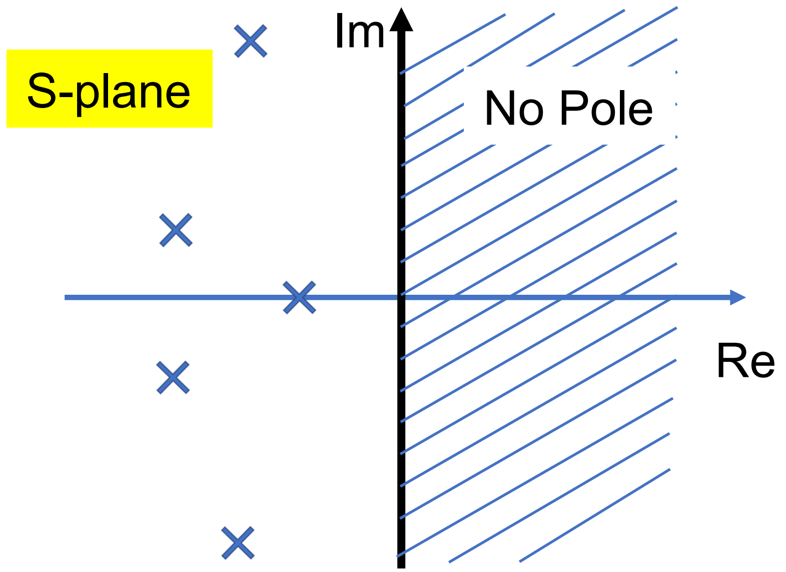
1.Strictly Proper, that is, a zero exists at $s = \infty,\quad \lim_{s\to\infty} G(s) = 0$

$$G(s) = \frac{N(s)}{D(s)}$$

The order of polynomial D(s) is higher than N(s).

2. Analytic in the right hand plane, and

3. Continuous in $\mathrm{Re}[s] \geq 0$



Then, there exists a sequence $\{\bar{g}_k\}$ such that

$$G(s) = \sum_{k=1}^{\infty} \bar{g}_k \frac{\sqrt{2\bar{a}}}{s+\bar{a}} \left(\frac{s-\bar{a}}{s+\bar{a}} \right)^{k-1}$$

where $\bar{a} > 0$ is a positive constant, called a Laguerre pole.

Proof

Consider a complex number - to - complex number transformation

$$z = \frac{s+\bar{a}}{s-\bar{a}}, \quad s \mapsto z$$

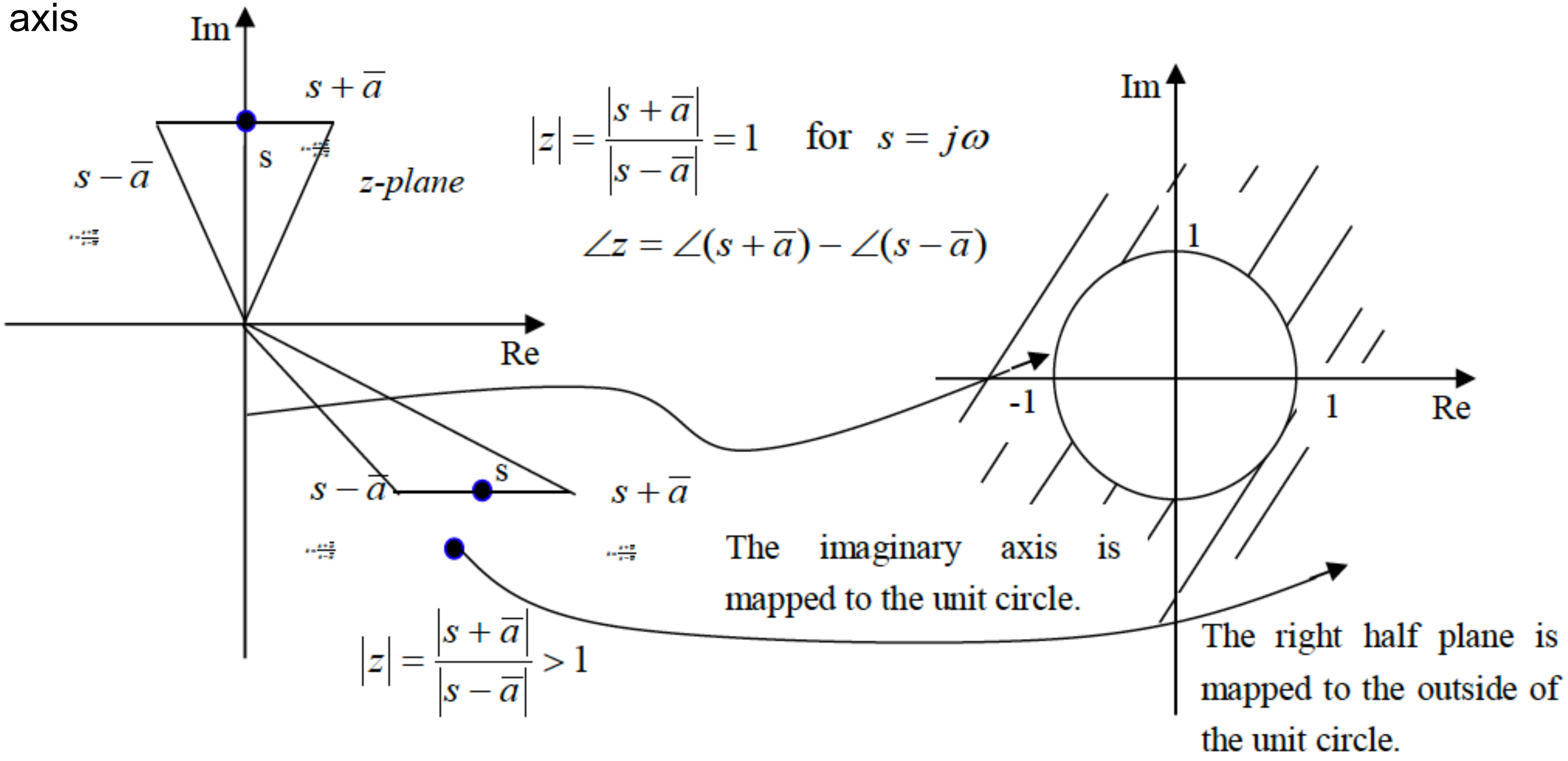
This is called a bilinear transform.

Pick s on the imaginary axis

$$|z| = \frac{|s+\bar{a}|}{|s-\bar{a}|} = 1$$

Pick s on the RHS

$$|z| = \frac{|s+\bar{a}|}{|s-\bar{a}|} > 1$$



This bilinear transformation preserves stability and phase.

The inverse transform of $z = \dfrac{s+\bar{a}}{s-\bar{a}}$,

$$(s-\bar{a})z = s+\bar{a}, \quad (z-1)s = (z+1)\bar{a}, \quad \therefore s = \frac{z+1}{z-1}\bar{a}$$

Substituting s by $\dfrac{z+1}{z-1}\bar{a}$ yields

$$G(s) = G\left(\frac{z+1}{z-1}\bar{a}\right) \triangleq \bar{G}(z)$$

Example 1

Obtain $\bar{G}(z)$ for the following G(s).

$$G(s) = \frac{1}{(s+1)(s+2)}$$

The inverse transform of $z = \dfrac{s+\bar{a}}{s-\bar{a}}$,

$$(s-\bar{a})z = s+\bar{a}, \quad (z-1)s = (z+1)\bar{a}, \quad \therefore s = \frac{z+1}{z-1}\bar{a}$$

Substituting s by $\dfrac{z+1}{z-1}\bar{a}$ yields

$$G(s) = G\left(\frac{z+1}{z-1}\bar{a}\right) \triangleq \bar{G}(z)$$

Example 1

Obtain $\bar{G}(z)$ for the following G(s).

$$G(s) = \frac{1}{(s+1)(s+2)} = \frac{1}{\left(\dfrac{z+1}{z-1}\bar{a}+1\right)\left(\dfrac{z+1}{z-1}\bar{a}+2\right)}$$

$$\bar{G}(z) = \frac{(z-1)^2}{[(\bar{a}+1)z+\bar{a}-1][(\bar{a}+2)z+\bar{a}-2]}$$

Proof of Theorem 1 continued

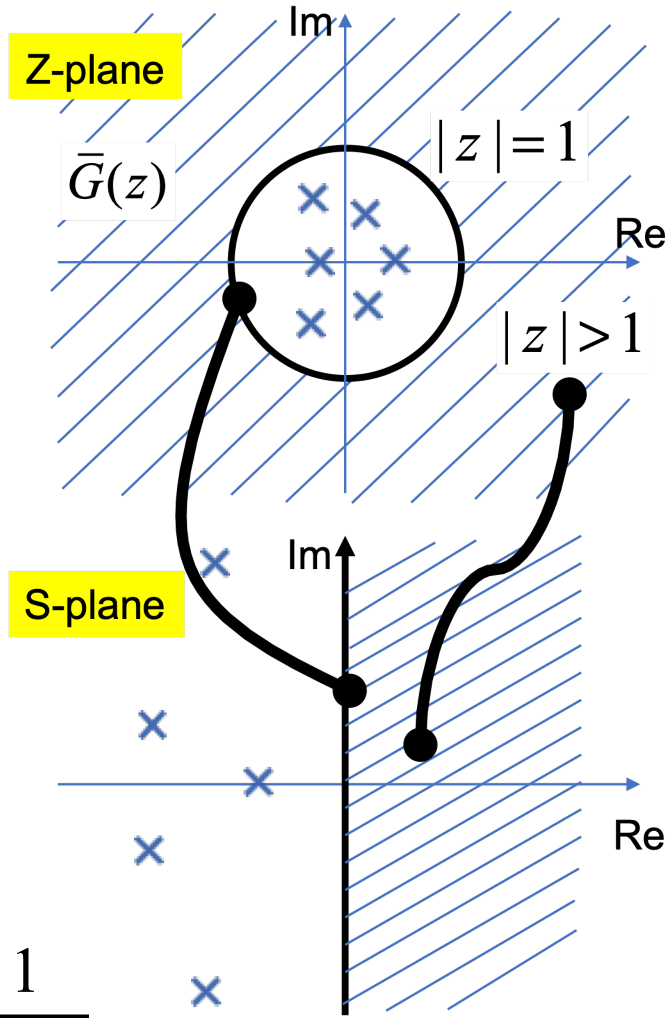
- From the assumption, G(s) is analytic in Re[s] > 0. Therefore, $\bar{G}(z)$ is analytic outside the unit circle.
- This implies that $\bar{G}(z)$ must be expressed as a Laurent Expansion

$$\exists\{g_k\} \quad \text{such that} \quad \bar{G}(z) = \sum_{k=1}^{\infty} g_k z^{-k}$$

- From the assumption, G(s) is strictly proper:

A zero exists at $s = \infty$, $\lim_{s\to\infty} G(s) = 0$

- How about $\bar{G}(z)$? Note that $\lim_{s\to\infty} z = \lim_{s\to\infty} \frac{s+\bar{a}}{s-\bar{a}} = 1$
- Namely, at z = 1, $\bar{G}(z) = 0$. This implies that (z-1) must be a factor of $\bar{G}(z)$.



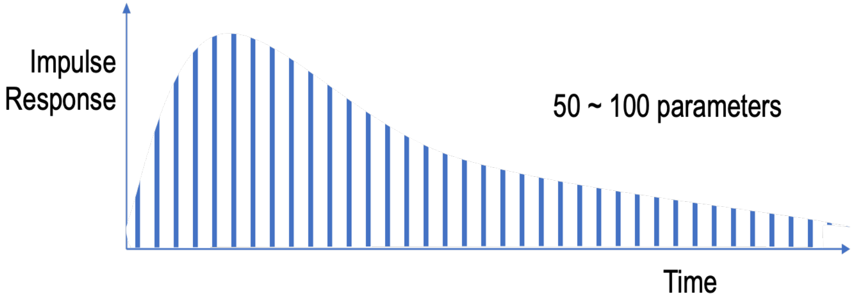
$$\bar{G}(z) = (z-1)\bar{G}'(z) = z(1-z^{-1})\bar{G}'(z) = z(1-z^{-1})\cdot\sum_{k=1}^{\infty}\bar{g}_k z^{-k}\frac{1}{\sqrt{2\bar{a}}}$$

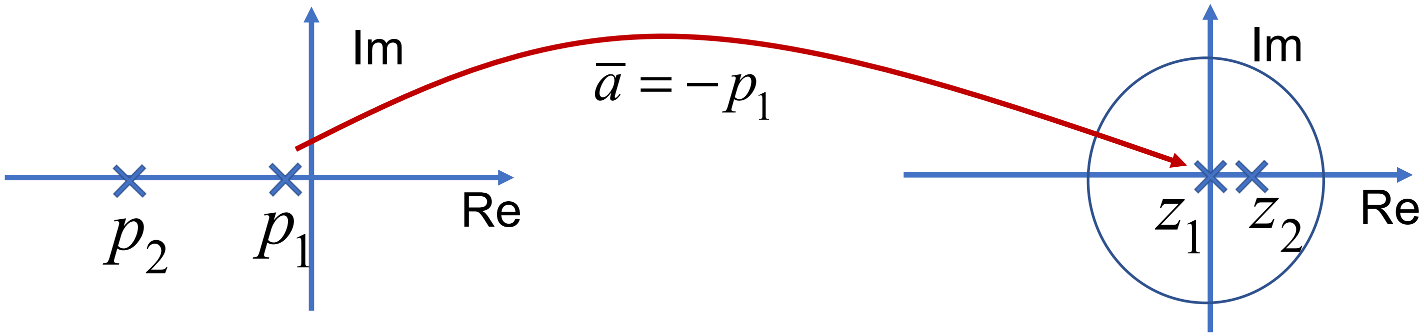
$$= \frac{1-z^{-1}}{\sqrt{2\bar{a}}}\sum_{k=1}^{\infty}\bar{g}_k\left(\frac{s-\bar{a}}{s+\bar{a}}\right)^{k-1} = \frac{\sqrt{2\bar{a}}}{s+\bar{a}}\sum_{k=1}^{\infty}\bar{g}_k\left(\frac{s-\bar{a}}{s+\bar{a}}\right)^{k-1}$$

Q.E.D.

The Catch

- Our objective is to reduce the order of FIR model; a slowly decaying pole prolongs the convergence of impulse response. Such a pole is located near the origin or the imaginary axis.

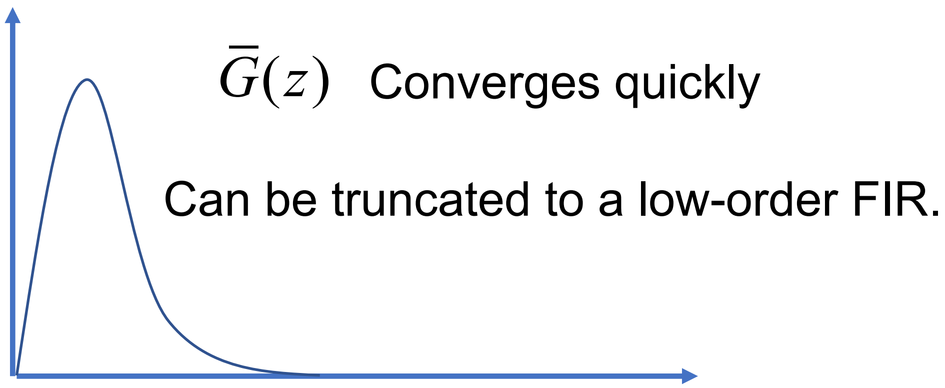




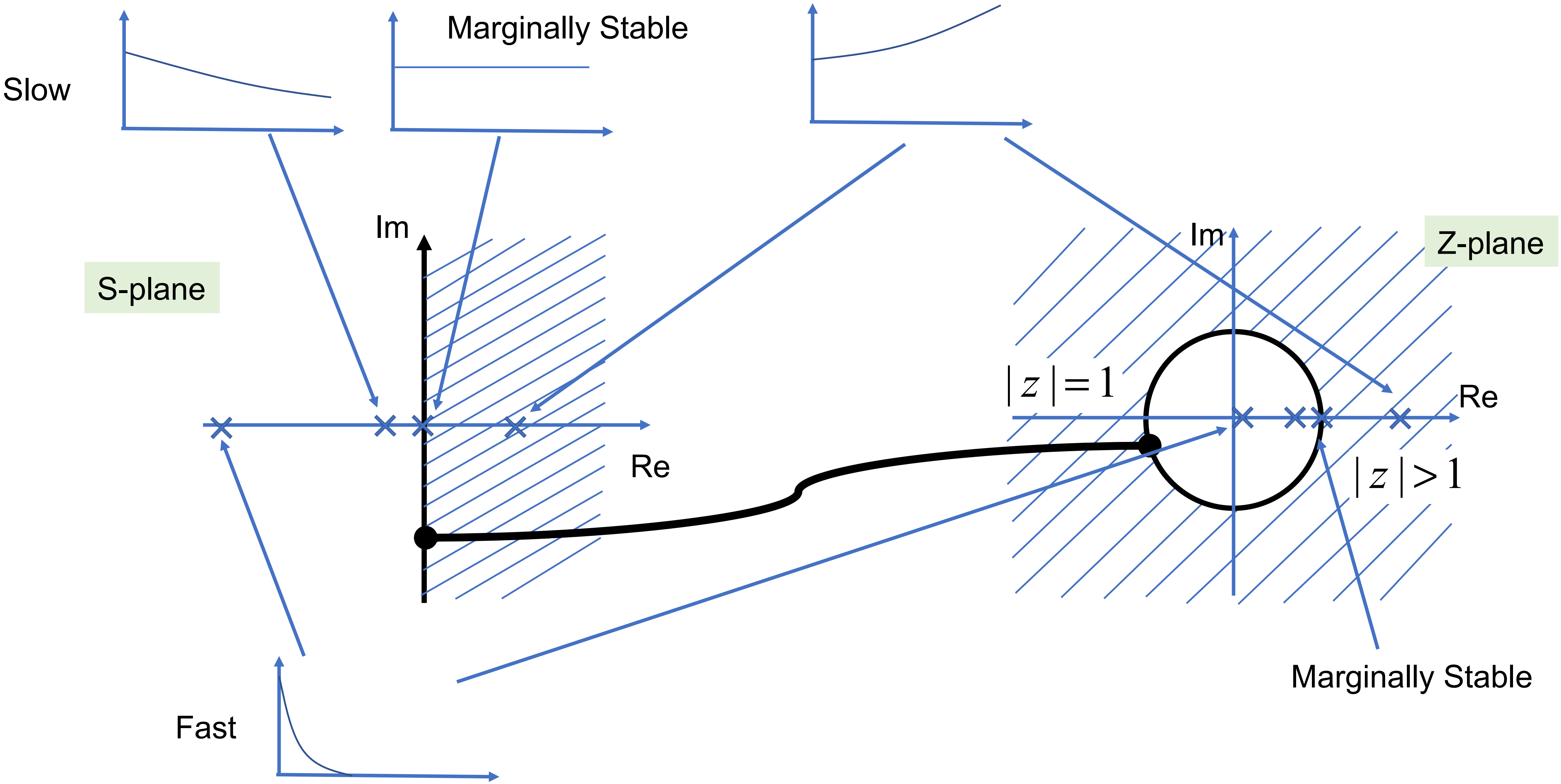
- The slow pole can be relocated by using the bilinear transformation. If we set the Laguerre pole at the slow pole: $\bar{a} = -p_1$

$$z = \frac{s+\bar{a}}{s-\bar{a}} \quad \Rightarrow \quad z_1 = z\Big|_{s=p_1} = \frac{s-p_1}{s+p_1}\Big|_{s=p_1} = 0$$

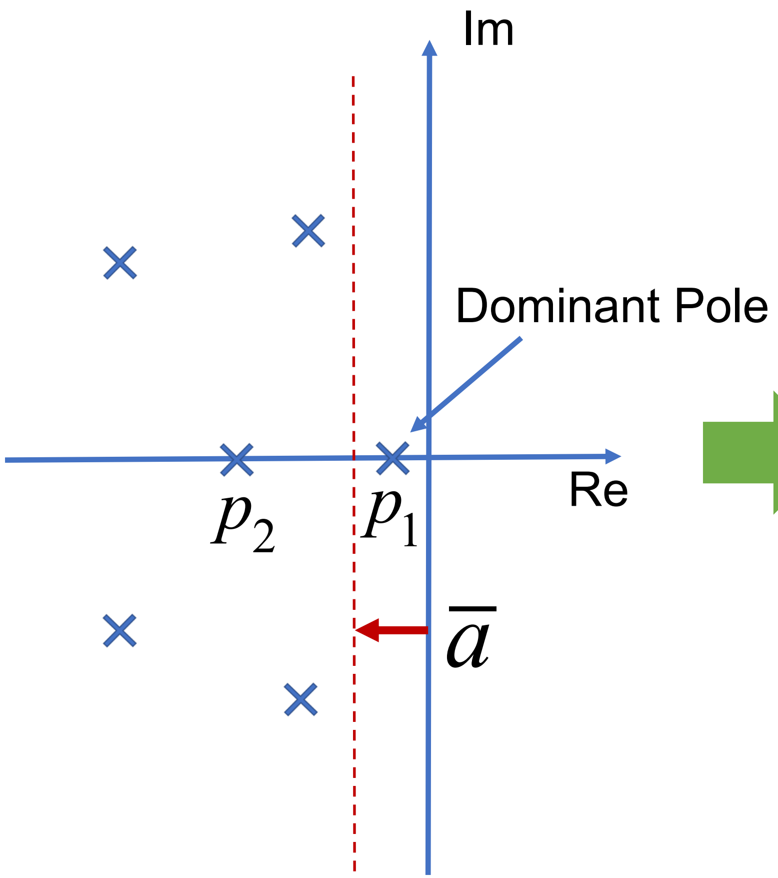
- With the slow pole being mapped to the origin in the z-plane, the system's impulse response in the z-plane converges instantaneously. Furthermore, if other poles in the s-plane are close to p_1, they are mapped to a region away from the unit circle in the z-plane.

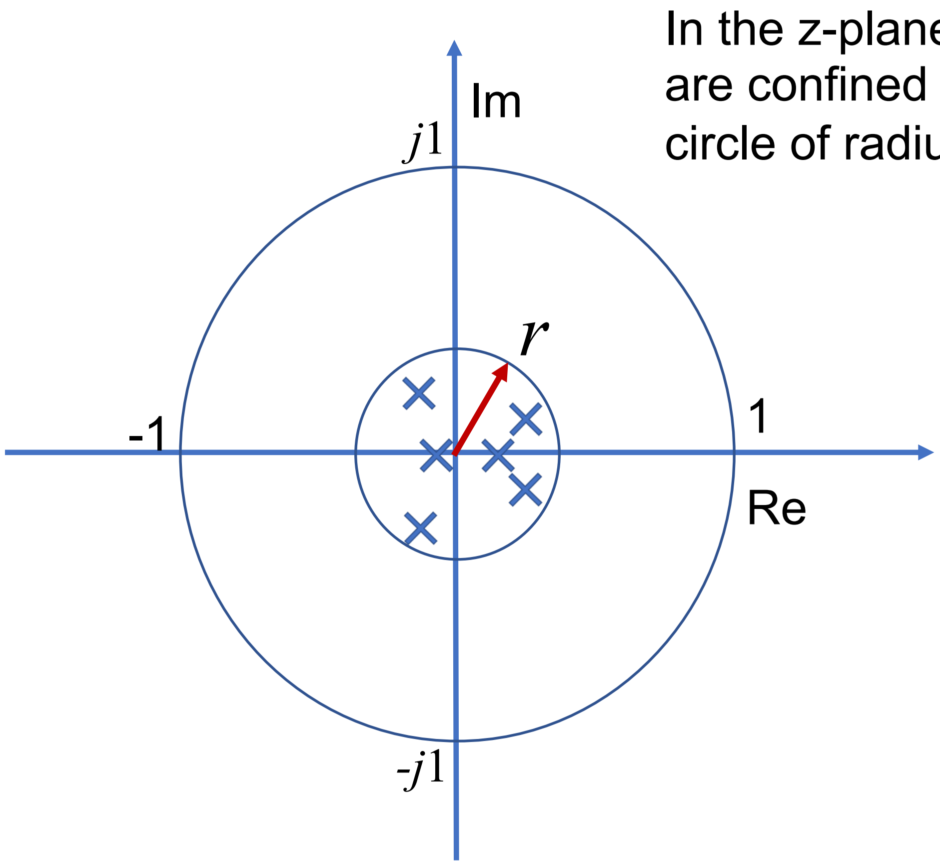


Review of UG Classical Control: Impulse Response



Applying the Laguerre Series Expansion to FIR model compression





In the z-plane, all the poles are confined within a small circle of radius r << 1.

If there are many poles are involved, we pick the Laguerre pole near the dominant pole, that is, the pole closest to the imaginary axis in the s-plane.

$\bar{G}(z)$ can be approximated to a low-order FIR.

In s, we can write

$$G(s) \cong \sum_{k=1}^{n} \bar{g}_k \frac{\sqrt{2\bar{a}}}{s+\bar{a}} \left(\frac{s-\bar{a}}{s+\bar{a}} \right)^{k-1}$$

where n is small.

Example 2

Compress the impulse response of the following transfer function using Laguerre Series Expansion.

$$G(s) = \frac{1}{(s+1)^2}$$

The poles in the s-plane are: $s = -1$, repeated poles.

Using the inverse bilinear transformation,

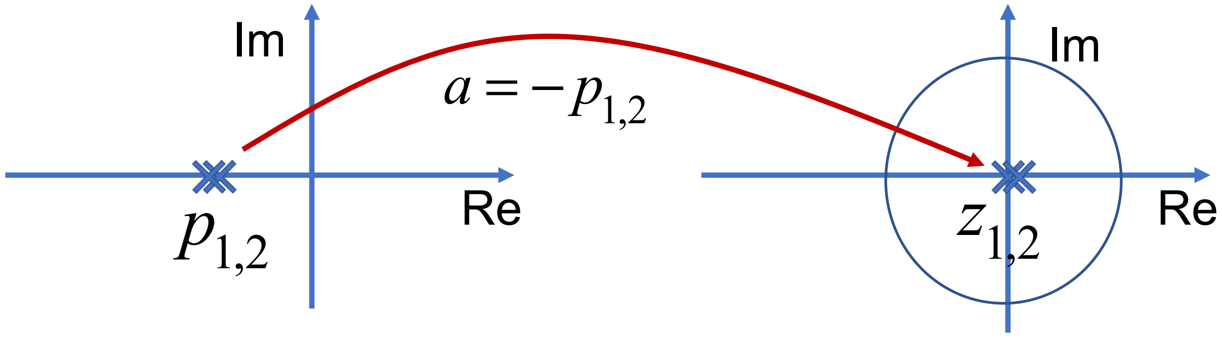
$$s = a\frac{z+1}{z-1}$$

the transfer function in the z-plane is given by

$$\bar{G}(z) = \frac{(z-1)^2}{[(a+1)z + (a-1)]^2}$$

The poles in the z-space are: $z_{1,2} = -\frac{a-1}{a+1}$

Repeated Poles



With $a = 1$,

$$\bar{G}(z) = \frac{(z-1)^2}{[2z]^2} = \frac{z^2 - 2z + 1}{4z^2}$$

$$= \frac{1}{4} - \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2}$$ Finite!

Setting the Laguerre pole at $a = 1$, we can shift both poles to

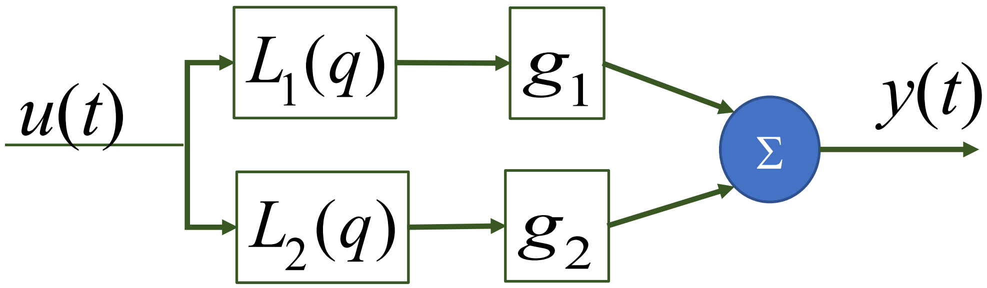
$$z_{1,2} = 0$$

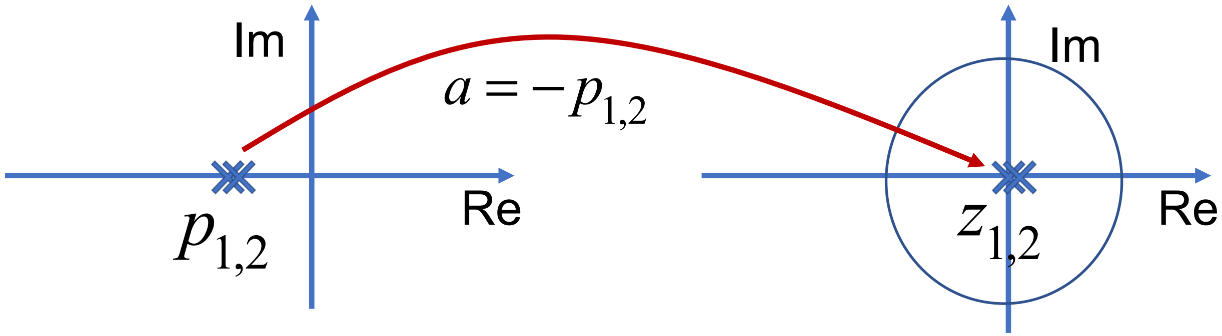
Converges at the 2nd order

Example 2 continued

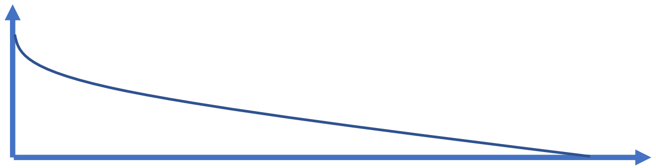
The Laguerre Series Expansion converges at $n = 2$.

$$\bar{G}(z) = \frac{1}{4} - \frac{1}{2} z^{-1} + \frac{1}{4} z^{-2} \Longleftrightarrow G(s) = \sum_{k=1}^{2} \bar{g}_k \frac{\sqrt{2\bar{a}}}{s+\bar{a}} \left(\frac{s-\bar{a}}{s+\bar{a}} \right)^{k-1} = \bar{g}_1 \frac{\sqrt{2\bar{a}}}{s+\bar{a}} + \bar{g}_2 \frac{\sqrt{2\bar{a}}}{s+\bar{a}} \frac{s-\bar{a}}{s+\bar{a}}$$





$$L_1(q) = \frac{\sqrt{2\bar{a}}}{s+\bar{a}}$$



$$L_2(q) = \frac{\sqrt{2\bar{a}}}{s+\bar{a}} \frac{s-\bar{a}}{s+\bar{a}}$$

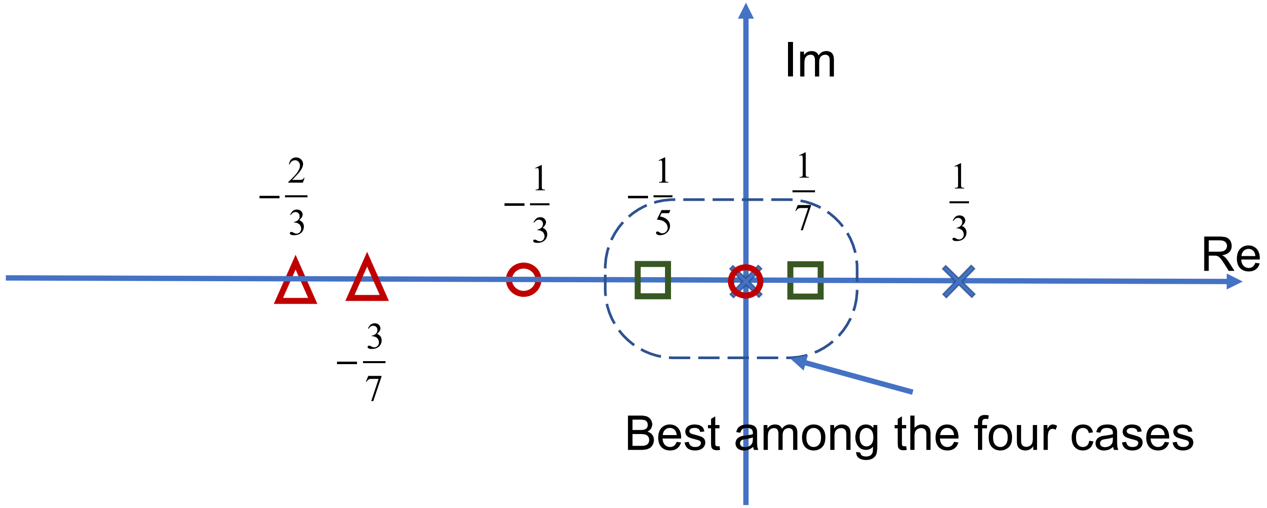
Just find 2 parameters.

$$G(s) = \frac{1}{(s+1)^2} \quad \text{with} \quad \bar{g}_1 = \frac{\sqrt{2}}{4}, \bar{g}_2 = -\frac{\sqrt{2}}{4}$$

Example 3

Find the Laguerre pole that effectively compresses the impulse response of the following transfer function

$$G(s) = \frac{1}{(s+1)(s+2)}$$



The poles of $\bar{G}(z)$

$$z_1 = -\frac{a-1}{a+1}, \quad z_2 = -\frac{a-2}{a+2}$$

Case	a	z_1	z_2	legend
1	1	0	1/3	×
2	1.5	-1/5	1/7	○
3	2	-1/3	0	□
4	5	-2/3	-3/7	△

$$a^o = \arg\min_{a>0}\left(\max\left[\left|\frac{a-1}{a+1}\right|, \left|\frac{a-2}{a+2}\right|\right]\right)$$

15.3 Discrete-Time Laguerre Series Expansion

[Theorem 15.2]

Assume that a Z-transform G(z) is

- Strictly proper $G(\infty) = 0$
- Analytic in $|z| > 1$ RHP
- Continuous in $|z| \geq 1$

Then

$$G(z) = \sum_{k=1}^{\infty} \bar{g}_k \frac{K}{z-a}\left(\frac{1-az}{z-a}\right)^{k-1}$$

where -1 < a < 1 and

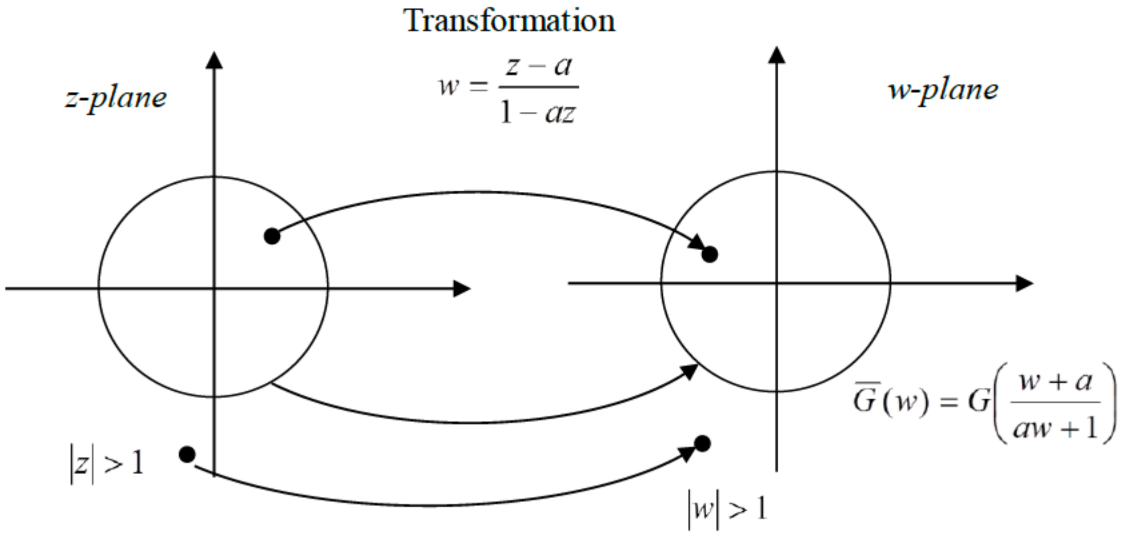
$$K = \sqrt{(1-a^2)T} \qquad T = \text{sampling Interval}$$

Proof

Consider the bilinear transformation:

$$w = \frac{z-a}{1-az}$$

$$w - azw = z - a \qquad w + a = z(aw+1) \quad \text{therefore,} \quad z = \frac{w+a}{aw+1}$$



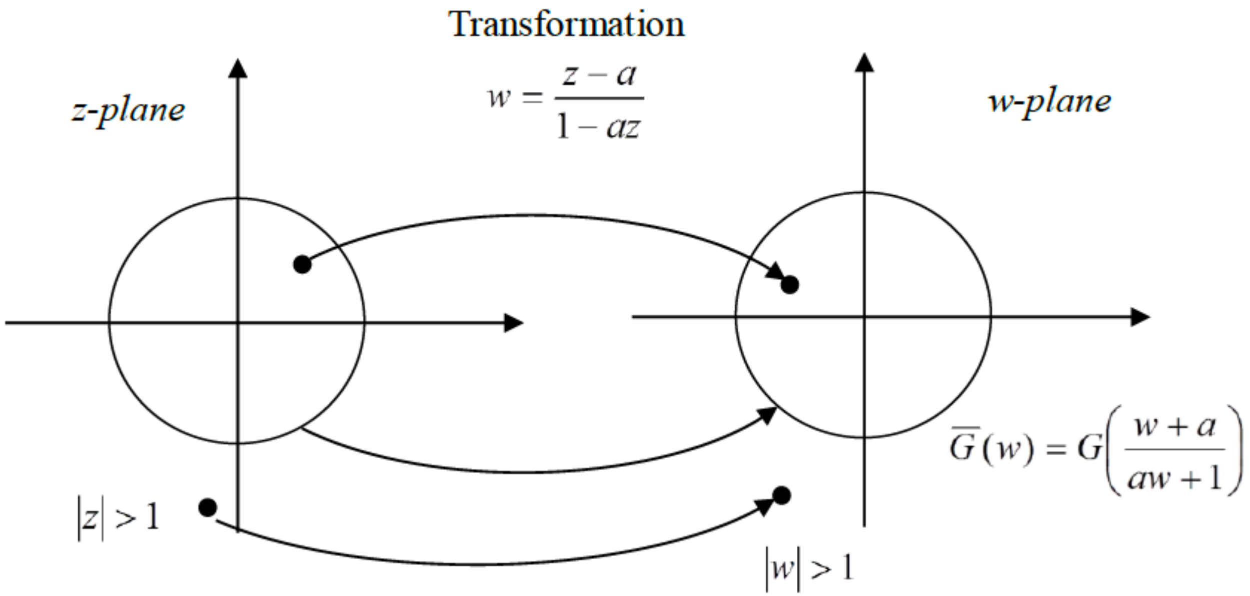
$$\overline{G}(w) = G\left(\frac{w+a}{aw+1}\right) \text{ is analytic in } |w| > 1, \text{ and is proper } G(\infty) = 0$$

$$\lim_{z\to\infty} w = -\frac{1}{a} \qquad \overline{G}(-\frac{1}{a}) = 0$$

$$\overline{G}(w) = \frac{T}{K}(a + w^{-1})\sum_{k=1}^{\infty} \overline{g}_k w^{-(k-1)}$$

$$G(z) = \overline{G}\left(\frac{z-a}{1-az}\right) = \frac{T}{K}(a + \frac{1-az}{z-a})\sum_{k=1}^{\infty} \overline{g}_k \left(\frac{z-a}{1-az}\right)^{-(k-1)}$$

$$\therefore G(z) = \sum_{k=1}^{\infty} \overline{g}_k \frac{K}{z-a}\left(\frac{1-az}{z-a}\right)^{k-1}$$



Now we can write

$$y(t) = G(q)u(t)$$

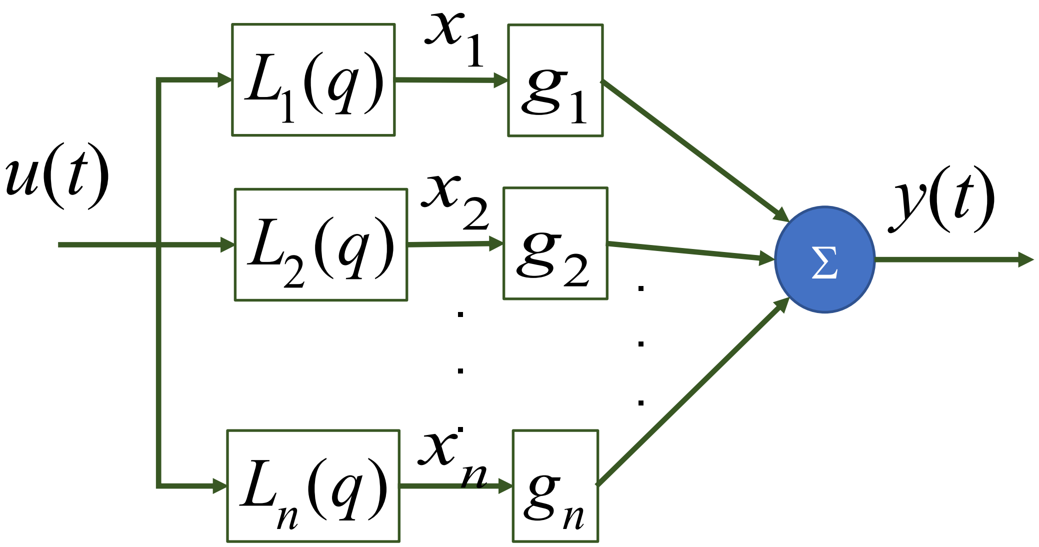
$$= \sum_{k=1}^{n} \overline{g}_k \frac{K}{q-a}\left(\frac{1-aq}{q-a}\right)^{k-1} u(t) = \sum_{k=1}^{n} \overline{g}_k L_k(q)u(t)$$

Now we can write

$$y(t) = G(q)u(t)$$

$$= \sum_{k=1}^{n} \bar{g}_k \frac{K}{q-a}\left(\frac{1-aq}{q-a}\right)^{k-1} u(t) = \sum_{k=1}^{n} \bar{g}_k L_k(q)u(t)$$

$$x_k = L_k(q)u(t), \quad k = 1, \cdots, n$$



Furthermore, $x_k(t)$ can be computed recursively.

$$x_1(t) = L_1(q)u(t) = \frac{K}{q-a}u(t) = \frac{Kq^{-1}}{1-aq^{-1}}u(t)$$

$$x_1(t) - ax_1(t-1) = Ku(t-1)$$

$$x_1(t) = ax_1(t-1) + Ku(t-1)$$

$$x_2(t) = \frac{1-aq}{q-a}x_1(t) = \frac{q^{-1}-a}{1-aq^{-1}}x_1(t)$$

$$x_2(t) - ax_2(t-1) = x_1(t-1) - ax_1(t)$$

$$x_2(t) = ax_2(t-1) + x_1(t-1) - ax_1(t)$$

$$x_k(t) = ax_k(t-1) + x_{k-1}(t-1) - ax_{k-1}(t)$$

Recursive Filters