2.160 Identification, Estimation, and Learning

Part 3 Linear System Identification

Lecture 18

Subspace Methods for System Identification: MOESP and N4SID

moesp(y,u,d)
n4sid (data,nx)

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Summary of the previous results on Subspace Methods -1

- System parameter matrices, A, B, C, and D, are linearly involved in state space representation; Once states are obtained from input-output data, it is a simple least squares estimate problem.

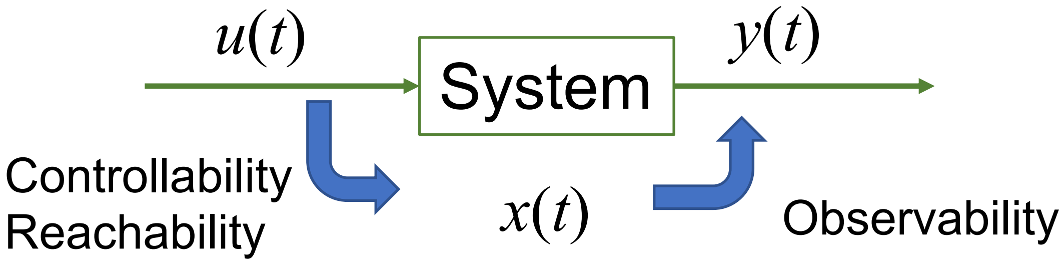
$$\underbrace{\begin{pmatrix} x(t+1) \\ y(t) \end{pmatrix}}_{Y(t)} = \underbrace{\begin{pmatrix} A & B \\ C & D \end{pmatrix}}_{\Theta} \underbrace{\begin{pmatrix} x(t) \\ u(t) \end{pmatrix}}_{\varphi(t)}$$

- The Hankel matrix of impulse response is the product of extended observability and reachability matrices.

- Singular-Value Decomposition of Hankel matrix H gives observability and reachability matrices,

$$H = \mathcal{O}_k \mathcal{C}_k = U_1 \Sigma_1 V_1^T \Rightarrow \mathcal{O}_k = U_1 \Sigma_1^{1/2} \quad \mathcal{C}_k = \Sigma_1^{1/2} V_1^T$$

from which B and C, and then A can be determined.



$\mathcal{O}_k = U_1 \Sigma_1^{1/2}$ Balanced Realization $\mathcal{C}_k = \Sigma_1^{1/2} V_1^T$

$$H = \begin{pmatrix} G_1 & G_2 & \cdots & G_k \\ G_2 & G_3 & & G_{k+1} \\ \vdots & & \ddots & \vdots \\ G_k & G_{k+1} & \cdots & G_{2k-1} \end{pmatrix} = \begin{pmatrix} CB & CAB & \cdots & CA^{k-1}B \\ CAB & CA^2B & & CA^kB \\ \vdots & & \ddots & \vdots \\ CA^{k-1}B & CA^kB & \cdots & CA^{2k-2}B \end{pmatrix} = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{k-1} \end{pmatrix} \begin{pmatrix} B & AB & \cdots & A^{k-1}B \end{pmatrix} = \mathcal{O}_k \mathcal{C}_k$$

Summary of the previous results on Subspace Methods -2

- Input output data in the Hankel Matrix form
  - Input Data Matrix
  - Output Data Matrix
  - Data Matrix

$$U_{0|k-1} = \begin{pmatrix} u(0) & u(1) & \cdots & u(N-1) \\ u(1) & u(2) & & u(N) \\ \vdots & & \ddots & \vdots \\ u(k-1) & u(k) & \cdots & u(k+N-2) \end{pmatrix}$$

$$Y_{0|k-1} = \begin{pmatrix} y(0) & y(1) & \cdots & y(N-1) \\ y(1) & y(2) & & y(N) \\ \vdots & & \ddots & \vdots \\ y(k-1) & y(k) & \cdots & y(k+N-2) \end{pmatrix}$$

$$W_{0|k-1} = \begin{pmatrix} U_{0|k-1} \\ Y_{0|k-1} \end{pmatrix}$$

- Zero-input response reveals the impulse response Hankel matrix that is the product of $\mathcal{O}_k$ and $\mathcal{C}_k$.

Example 18-1

$$\begin{pmatrix} U_{0|3} \\ Y_{0|3} \end{pmatrix} = \left(\begin{array}{cccc:cccc} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & g_0 & g_1 & g_2 & g_3 & g_4 \\ 0 & 0 & g_0 & g_1 & g_2 & g_3 & g_4 & g_5 \\ 0 & g_0 & g_1 & g_2 & g_3 & g_4 & g_5 & g_6 \\ g_0 & g_1 & g_2 & g_3 & g_4 & g_5 & g_6 & g_7 \end{array}\right)$$

Zero input

This block is the Hankel matrix $H_{4,4}$

- Zero-input response can be created from arbitrary input-output data satisfying the 3 assumptions through column manipulation of the Hankel data matrix $W_{0|k-1}$.

$$\begin{pmatrix} \boldsymbol{u}_k(0) \\ \boldsymbol{y}_k(0) \end{pmatrix} = \begin{pmatrix} U_{0|k-1} \\ Y_{0|k-1} \end{pmatrix} \zeta, \quad W_{0|k-1} = \begin{pmatrix} U_{0|k-1} \\ Y_{0|k-1} \end{pmatrix}$$

Summary of the previous results on Subspace Methods -3

- Zero-input response can be created from arbitrary input-output data through column manipulation of the Hankel data matrix $W_{0|k-1}$, if it meets the 3 assumptions.

$$
\underbrace{\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ y^*(0) \\ y^*(1) \\ \vdots \\ y^*(k-1) \end{pmatrix}}_{\text{Zero-Input Response}}
=
\underbrace{\begin{pmatrix}
u(0) & u(1) & \cdots & u(N-1) \\
u(1) & u(2) & & u(N) \\
\vdots & \ddots & & \vdots \\
u(k-1) & u(k) & \cdots & u(k+N-2) \\
y(0) & y(1) & \cdots & y(N-1) \\
y(1) & y(2) & \ddots & y(N) \\
\vdots & & & \vdots \\
y(k-1) & y(k) & \cdots & y(k+N-2)
\end{pmatrix}}_{\text{Data Matrix}}
\begin{pmatrix} \zeta^1 \\ \zeta^2 \\ \zeta^3 \\ \vdots \\ \zeta^i \\ \vdots \\ \vdots \\ \zeta^N \end{pmatrix}
=
\underbrace{\begin{pmatrix} u(0) \\ u(1) \\ \vdots \\ u(k-1) \\ y(0) \\ y(1) \\ \vdots \\ y(k-1) \end{pmatrix}\zeta^1
+
\begin{pmatrix} u(1) \\ u(2) \\ \vdots \\ u(k) \\ y(1) \\ y(2) \\ \vdots \\ y(k) \end{pmatrix}\zeta^2
+\cdots+
\begin{pmatrix} u(N-1) \\ u(N) \\ \vdots \\ u(k+N-2) \\ y(N-1) \\ y(N) \\ \vdots \\ y(k+N-2) \end{pmatrix}\zeta^N}_{\text{A linear combination of column vectors of the data matrix}}
$$

Collective Input-Output Hankel Expression

- From state and measurement equations,

$$y(t) = Cx(t) + Du(t)$$

$$y(t+1) = CAx(t) + CBu(t) + Du(t+1)$$

$$y(t+2) = CA^2x(t) + CABu(t) + CBu(t+1) + Du(t+2)$$

$$\vdots \qquad\qquad \vdots$$

$$x(t+1) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t)$$

- These equations can be written collectively,

$$\underbrace{\begin{pmatrix} y(t) \\ y(t+1) \\ \vdots \\ y(t+k-1) \end{pmatrix}}_{\boldsymbol{y}_k(t) \quad pk\times 1} = \underbrace{\begin{pmatrix} C \\ CA \\ \vdots \\ CA^{k-1} \end{pmatrix}}_{O_k \quad pk\times n} x(t) + \underbrace{\begin{pmatrix} D & 0 & \cdots & 0 \\ CB & D & & \vdots \\ \vdots & & \ddots & 0 \\ CA^{k-2}B & \cdots & CB & D \end{pmatrix}}_{\Psi_k \quad pk\times mk} \underbrace{\begin{pmatrix} u(t) \\ u(t+1) \\ \vdots \\ u(t+k-1) \end{pmatrix}}_{\boldsymbol{u}_k(t) \quad mk\times 1}$$

- Or, succinctly,

$$\boldsymbol{y}_k(t) = O_k x(t) + \Psi_k \boldsymbol{u}_k(t)$$

This matrix is a block Toeplitz matrix.

Collective Input-Output Hankel Expression

- Note that concatenating $\boldsymbol{y}_k(0) \quad \boldsymbol{y}_k(1) \quad \cdots \quad \boldsymbol{y}_k(N-1)$ yields the following block Hankel output matrix,

$$Y_{0|k-1} = \begin{pmatrix} \boldsymbol{y}_k(0) & \boldsymbol{y}_k(1) & \cdots & \boldsymbol{y}_k(N-1) \end{pmatrix}$$

Similarly, $U_{0|k-1} = \begin{pmatrix} \boldsymbol{u}_k(0) & \boldsymbol{u}_k(1) & \cdots & \boldsymbol{u}_k(N-1) \end{pmatrix}$

Also, we define $X_0 \triangleq \begin{pmatrix} x(0) & x(1) & \cdots & x(N-1) \end{pmatrix}$

- The input-output relationship, $\boldsymbol{y}_k(t) = O_k x(t) + \Psi_k \boldsymbol{u}_k(t)$, can be expanded to the block Hankel form,

$$Y_{0|k-1} = O_k X_0 + \Psi_k U_{0|k-1}$$

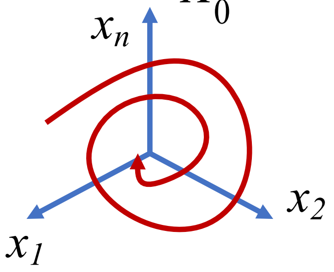
- This is a succinct expression of the following relationship.

$$\underbrace{\begin{pmatrix} y(0) & y(1) & \cdots & y(N-1) \\ y(1) & y(2) & & y(N) \\ \vdots & & \ddots & \vdots \\ y(k-1) & y(k) & \cdots & y(k+N-2) \end{pmatrix}}_{Y_{0|k-1}} = \underbrace{\begin{pmatrix} C \\ CA \\ \vdots \\ CA^{k-1} \end{pmatrix}}_{O_k} \underbrace{\begin{pmatrix} x(0) & x(1) & \cdots & x(N-1) \end{pmatrix}}_{X_0} + \underbrace{\begin{pmatrix} D & 0 & \cdots & 0 \\ CB & D & & \vdots \\ \vdots & & \ddots & 0 \\ CA^{k-2}B & \cdots & CB & D \end{pmatrix}}_{\Psi_k} \underbrace{\begin{pmatrix} u(0) & u(1) & \cdots & u(N-1) \\ u(1) & u(2) & & u(N) \\ \vdots & & \ddots & \vdots \\ u(k-1) & u(k) & \cdots & u(k+N-2) \end{pmatrix}}_{U_{0|k-1}}$$

Three Assumptions on Data

- For constructing subspace identification algorithms, we have to make three key assumptions on data.

$$\underbrace{\begin{pmatrix} y(0) & y(1) & \cdots & y(N-1) \\ y(1) & y(2) & & y(N) \\ \vdots & & \ddots & \vdots \\ y(k-1) & y(k) & \cdots & y(k+N-2) \end{pmatrix}}_{Y_{0|k-1} \quad pk \times N} = O_k \underbrace{\begin{pmatrix} x(0) & x(1) & \cdots & x(N-1) \end{pmatrix}}_{X_0} + \Psi_k \underbrace{\begin{pmatrix} u(0) & u(1) & \cdots & u(N-1) \\ u(1) & u(2) & & u(N) \\ \vdots & & \ddots & \vdots \\ u(k-1) & u(k) & \cdots & u(k+N-2) \end{pmatrix}}_{U_{0|k-1} \quad mk \times N}$$



- Assumption A-1: rank $X_0 = n$.

  The state vector is sufficiently excited, or the system is reachable.

- Assumption A-2: $\operatorname{rank} U_{0|k-1} = mk$

  The input sequence is persistently exciting of order k.

Recall $\sum \varphi(t)\varphi^T(t) = \text{(full rank)}$

$$\begin{pmatrix} \varphi(0) & \cdots & \varphi(N-1) \end{pmatrix} \begin{pmatrix} \varphi^T(0) \\ \vdots \\ \varphi^T(N-1) \end{pmatrix}$$

$$\Updownarrow$$

$$\begin{pmatrix} u(0) & \cdots & u(N-1) \end{pmatrix}$$

- Assumption A-3: $\operatorname{rank} \begin{pmatrix} U_{0|k-1} \\ X_0 \end{pmatrix} = mk + n$

  X_0 and $U_{0|k-1}$ are not collinear. No linear state feedback: $\cancel{u = Kx}$.

  In other words, the spaces spanned by the input matrix and the state matrix do not intersect.

  $\operatorname{span} X_0 \cap \operatorname{span} U_{0|k-1} = \{\phi\}$ Experiments should not be taken with linear state feedback, $u = Kx$.

LQ Decomposition

- Of particular interest is the input-output pair, $\boldsymbol{u}_k(0)$ and $\boldsymbol{y}_k(0)$, corresponding to zero-input response:

$$\begin{pmatrix} \boldsymbol{u}_k(0) \\ \boldsymbol{y}_k(0) \end{pmatrix} = \begin{pmatrix} 0 \\ \boldsymbol{y}^*_k(0) \end{pmatrix} = \begin{pmatrix} U_{0|k-1} \\ Y_{0|k-1} \end{pmatrix} \zeta$$

Recall Example 18-1.

Note that such an input-output pair can be created by a linear combination of the column vectors of the data matrix.

- We can find multiple zero-input responses by using different vectors ζ.
- Under the 3 assumptions on the data matrix, $(km+kp)$ linearly independent vectors ζ can produce zero-input responses.

Zero-input response

$$\begin{pmatrix} U_{0|k-1} \\ Y_{0|k-1} \end{pmatrix} \underbrace{\begin{pmatrix} \zeta_1 & \zeta_2 & \cdots & \zeta_{km+kp} \end{pmatrix}}_{Q=(Q_1, Q_2)} = \begin{pmatrix} L_{11} & \boxed{\begin{matrix} 0 \\ L_{22} \end{matrix}} \\ L_{21} & \end{pmatrix}$$

where $\zeta_1, \zeta_2, \ldots$ can be made orthogonal through the Householder transformation. These orthogonal vectors constitute orthogonal matrix Q.

- This can be achieved by column operations, but an effective algorithm, called ***LQ Decomposition***, exists to transform the data matrix (a rectangular matrix) to a block lower triangular matrix. Namely,

$$\begin{pmatrix} U_{0|k-1} \\ Y_{0|k-1} \end{pmatrix} = \begin{pmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} Q_1^T \\ Q_2^T \end{pmatrix}$$

where $L_{11} \in \Re^{km \times km}$ and $L_{22} \in \Re^{pm \times pm}$ are lower-triangular.

Note $Q^T Q = \begin{pmatrix} Q_1^T \\ Q_2^T \end{pmatrix} \begin{pmatrix} Q_1 & Q_2 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$

QR Decomposition

- LQ Decomposition is the transpose of so-called "QR Decomposition". An arbitrary rectangular matrix $A \in \Re^{m\times n}$ can be decomposed to an orthonormal matrix Q and an upper triangular matrix in the following form:

$$A = QR = \begin{pmatrix} \underset{\overset{\longleftrightarrow}{n}}{Q_1} & \underset{\overset{\longleftrightarrow}{m-n}}{Q_2} \end{pmatrix} \begin{pmatrix} R_1 \\ 0 \end{pmatrix} \qquad m \geq n \qquad R_1 = \begin{pmatrix} * & * & \cdots & * \\ 0 & * & * & \vdots \\ \vdots & 0 & * & * \\ 0 & \cdots & 0 & * \end{pmatrix} \in \Re^{n\times n}$$

- Matrix Q consists of unit-length column vectors that are orthogonal to each other.

$$Q^T Q = \begin{pmatrix} Q_1{}^T \\ Q_2{}^T \end{pmatrix} \begin{pmatrix} Q_1 & Q_2 \end{pmatrix} = \begin{pmatrix} Q_1{}^T Q_1 & Q_1{}^T Q_2 \\ Q_2{}^T Q_1 & Q_2{}^T Q_2 \end{pmatrix} \begin{pmatrix} I_n & 0 \\ 0 & I_{m-n} \end{pmatrix} = I_m$$

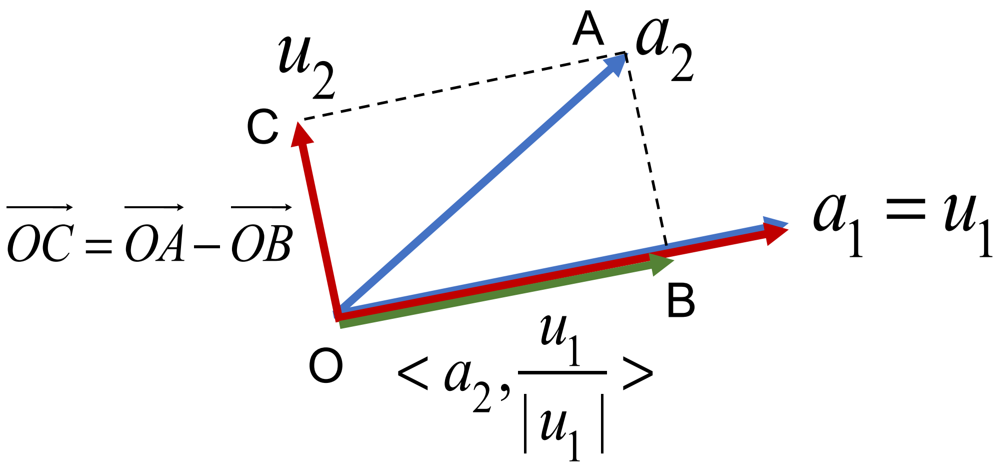
- MATLAB code: $\mathrm{qr(A)}$, $\mathrm{(Q, R) = qr(A)}$ returns an orthonormal matrix Q and an upper triangular matrix of the above form.
- There are effective algorithms to obtain the QR factorization of a rectangular matrix.
  - Gram-Schmidt procedure – numerically not stable
  - Householder Reflection – widely used method

QR Decomposition
= Transpose of LQ Decomposition

$$A = QR = \begin{pmatrix} Q_1 & Q_2 \end{pmatrix} \begin{pmatrix} R_1 \\ 0 \end{pmatrix} \qquad m \geq n \qquad R_1 = \begin{pmatrix} * & * & \cdots & * \\ 0 & * & * & \vdots \\ \vdots & 0 & * & * \\ 0 & \cdots & 0 & * \end{pmatrix} \in \Re^{n \times n}$$

- QR Decomposition Algorithm based on the Gram-Schmidt orthogonalization method

$$A = \begin{pmatrix} a_1 & a_2 & \cdots & a_m \end{pmatrix}$$



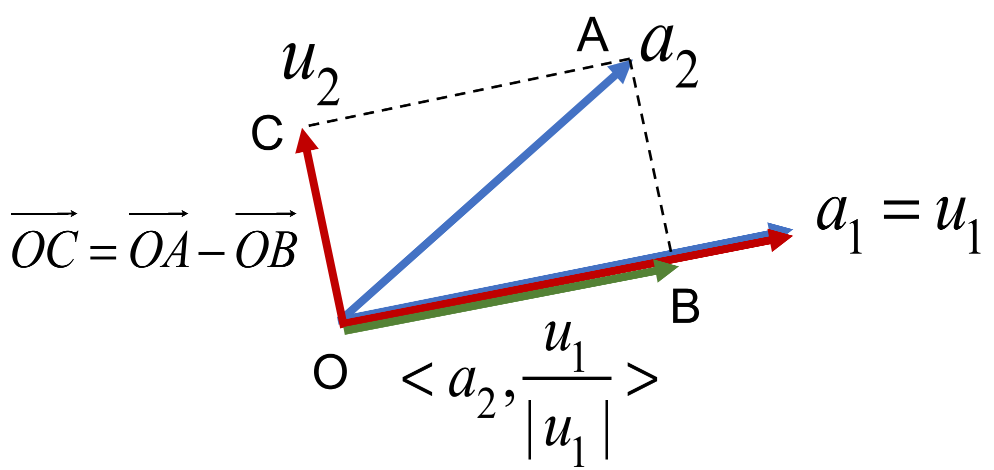
$$\mathbf{u}_1 = \mathbf{a}_1, \qquad \mathbf{e}_1 = \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|}$$

$$\mathbf{u}_2 = \mathbf{a}_2 - \mathrm{proj}_{\mathbf{u}_1}\, \mathbf{a}_2, \quad \overrightarrow{OC} = \overrightarrow{OA} - \overrightarrow{OB} \qquad \mathbf{e}_2 = \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|}$$

$$\mathbf{u}_3 = \mathbf{a}_3 - \mathrm{proj}_{\mathbf{u}_1}\, \mathbf{a}_3 - \mathrm{proj}_{\mathbf{u}_2}\, \mathbf{a}_3, \qquad \mathbf{e}_3 = \frac{\mathbf{u}_3}{\|\mathbf{u}_3\|}$$

$$\vdots \qquad\qquad \vdots$$

$$\mathbf{u}_k = \mathbf{a}_k - \sum_{j=1}^{k-1} \mathrm{proj}_{\mathbf{u}_j}\, \mathbf{a}_k, \qquad \mathbf{e}_k = \frac{\mathbf{u}_k}{\|\mathbf{u}_k\|}$$



$$u_2 = a_2 - < a_2, \frac{u_1}{|u_1|} > \frac{u_1}{|u_1|} = \left(I - \frac{u_1 u_1^T}{|u_1|^2} \right) a_2$$

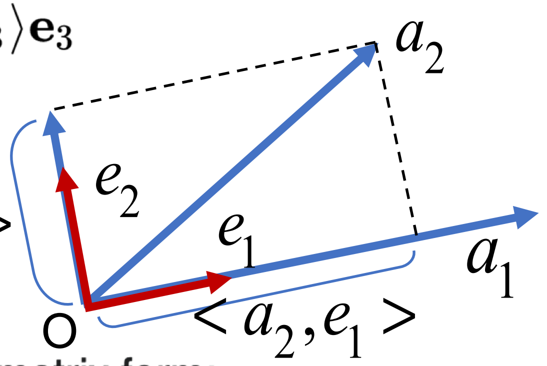
$$\begin{aligned}
\mathbf{u}_1 &= \mathbf{a}_1, \\
\mathbf{u}_2 &= \mathbf{a}_2 - \mathrm{proj}_{\mathbf{u}_1}\, \mathbf{a}_2, \\
\mathbf{u}_3 &= \mathbf{a}_3 - \mathrm{proj}_{\mathbf{u}_1}\, \mathbf{a}_3 - \mathrm{proj}_{\mathbf{u}_2}\, \mathbf{a}_3, \\
&\vdots \\
\mathbf{u}_k &= \mathbf{a}_k - \sum_{j=1}^{k-1} \mathrm{proj}_{\mathbf{u}_j}\, \mathbf{a}_k,
\end{aligned}$$

$$\boxed{\begin{aligned}
\mathbf{e}_1 &= \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} \\
\mathbf{e}_2 &= \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} \\
\mathbf{e}_3 &= \frac{\mathbf{u}_3}{\|\mathbf{u}_3\|} \\
&\vdots \\
\mathbf{e}_k &= \frac{\mathbf{u}_k}{\|\mathbf{u}_k\|}
\end{aligned}}$$

Orthonormal Basis

We can now express the $\mathbf{a}_i$s over our newly computed orthonormal basis:

$$\begin{aligned}
\mathbf{a}_1 &= \langle \mathbf{e}_1, \mathbf{a}_1 \rangle \mathbf{e}_1 \\
\mathbf{a}_2 &= \langle \mathbf{e}_1, \mathbf{a}_2 \rangle \mathbf{e}_1 + \langle \mathbf{e}_2, \mathbf{a}_2 \rangle \mathbf{e}_2 \\
\mathbf{a}_3 &= \langle \mathbf{e}_1, \mathbf{a}_3 \rangle \mathbf{e}_1 + \langle \mathbf{e}_2, \mathbf{a}_3 \rangle \mathbf{e}_2 + \langle \mathbf{e}_3, \mathbf{a}_3 \rangle \mathbf{e}_3 \\
&\vdots \\
\mathbf{a}_k &= \sum_{j=1}^{k} \langle \mathbf{e}_j, \mathbf{a}_k \rangle \mathbf{e}_j
\end{aligned}$$



where $\langle \mathbf{e}_i, \mathbf{a}_i \rangle = \|\mathbf{u}_i\|$. This can be written in matrix form:

$$A = QR \qquad A = \begin{pmatrix} a_1 & a_2 & \cdots & a_m \end{pmatrix}$$

where:

$$Q = [\mathbf{e}_1, \ldots, \mathbf{e}_n]$$

and

$$R = \begin{pmatrix} \langle \mathbf{e}_1, \mathbf{a}_1 \rangle & \langle \mathbf{e}_1, \mathbf{a}_2 \rangle & \langle \mathbf{e}_1, \mathbf{a}_3 \rangle & \ldots \\ 0 & \langle \mathbf{e}_2, \mathbf{a}_2 \rangle & \langle \mathbf{e}_2, \mathbf{a}_3 \rangle & \ldots \\ 0 & 0 & \langle \mathbf{e}_3, \mathbf{a}_3 \rangle & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

MOESP:
The Multivariable Output Error State sPace method

- From the LQ decomposition:

$$\begin{pmatrix} U_{0|k-1} \\ Y_{0|k-1} \end{pmatrix} = \begin{pmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} Q_1^T \\ Q_2^T \end{pmatrix} \Rightarrow \begin{array}{l} U_{0|k-1} = L_{11} Q_1^T \quad \text{and} \\ Y_{0|k-1} = L_{21} Q_1^T + L_{22} Q_2^T \end{array}$$

- Recall the collective input-output relationship using Hankel data matrices.

$$\underbrace{\begin{pmatrix} y(0) & y(1) & \cdots & y(N-1) \\ y(1) & y(2) & & y(N) \\ \vdots & & \ddots & \vdots \\ y(k-1) & y(k) & \cdots & y(k+N-2) \end{pmatrix}}_{Y_{0|k-1}} = \underbrace{\begin{pmatrix} C \\ CA \\ \vdots \\ CA^{k-1} \end{pmatrix}}_{O_k} \underbrace{\begin{pmatrix} x(0) & x(1) & \cdots & x(N-1) \end{pmatrix}}_{X_0} + \underbrace{\begin{pmatrix} D & 0 & \cdots & 0 \\ CB & D & & \vdots \\ \vdots & & \ddots & 0 \\ CA^{k-2}B & \cdots & CB & D \end{pmatrix}}_{\Psi_k} \underbrace{\begin{pmatrix} u(0) & u(1) & \cdots & u(N-1) \\ u(1) & u(2) & & u(N) \\ \vdots & & \ddots & \vdots \\ u(k-1) & u(k) & \cdots & u(k+N-2) \end{pmatrix}}_{U_{0|k-1}}$$

Or the succinct form $\quad Y_{0|k-1} = O_k X_0 + \Psi_k U_{0|k-1}$

- Substitution of the LQ decomposed relations yields

$$Y_{0|k-1} = O_k X_0 + \Psi_k L_{11} Q_1^T = L_{21} Q_1^T + L_{22} Q_2^T$$

MOESP (Continued)

- Post multiplying Q_2 to $O_k X_0 + \Psi_k L_{11} Q_1^T = L_{21} Q_1^T + L_{22} Q_2^T$ yields $O_k X_0 Q_2 = L_{22}$

  since $\begin{pmatrix} Q_1^T \\ Q_2^T \end{pmatrix} \begin{pmatrix} Q_1 & Q_2 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$

- Taking Singular-Value Decomposition of L_{22},

$$L_{22} = \begin{pmatrix} U_1 & U_2 \end{pmatrix} \begin{pmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V_1^T \\ V_2^T \end{pmatrix} = U_1 \Sigma_1 V_1^T \qquad \therefore O_k X_0 Q_2 = U_1 \Sigma_1 V_1^T$$

- Splitting this, we can find the observability matrix given by $O_k = U_1 \Sigma_1^{1/2}$

$$O_k = \begin{pmatrix} \boxed{C} \\ CA \\ \vdots \\ CA^{k-1} \end{pmatrix}$$

- The first block of the observability matrix is the matrix C: $C = O_k(1:p,:)$

- As before, the A matrix can be obtained from $O_k(1:p(k-1),:)A = O_k(p+1:pk,:)$

$$\therefore A = O_k(1:p(k-1),:)^{\#} O_k(p+1:pk,:)$$

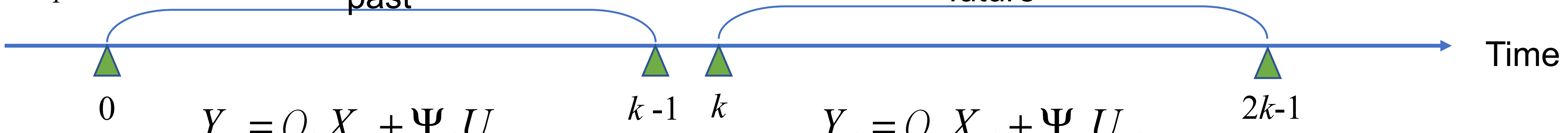
- Matrices B and D can also be determined from $O_k X_0 + \Psi_k L_{11} Q_1^T = L_{21} Q_1^T + L_{22} Q_2^T$ though computation is more tedious.

N4SID: (Read “Enforce It”.)
Numerical algorithm for Subspace State Space System Identification

- The N4SID method developed by Van Overschee and De Moor [1994] is one of the most prevailing algorithms for Subspace System Identification.
- This method, too, uses LQ Decomposition and Singular Value Decomposition. However, its data structure differs from the previous methods, and it applies a different geometric approach, called Oblique Projection.
- The following is the outline of the algorithm.

- Data structure:

$X_p \triangleq X_0$ $\quad U_p \triangleq U_{0|k-1}$ $\quad Y_p \triangleq Y_{0|k-1}$ $\quad X_f \triangleq X_k$ $\quad U_f \triangleq U_{k|2k-1}$ $\quad Y_f \triangleq Y_{k|2k-1}$



$$Y_p = O_k X_0 + \Psi_k U_p \qquad Y_f = O_k X_f + \Psi_k U_f$$

$$U_p \triangleq U_{0|k-1} = \begin{pmatrix} u(0) & u(1) & \cdots & u(N-1) \\ u(1) & u(2) & \cdots & u(N) \\ \vdots & \vdots & \ddots & \vdots \\ u(k-1) & u(k) & \cdots & u(k+N-2) \end{pmatrix} \qquad U_f \triangleq U_{k|2k-1} = \begin{pmatrix} u(k) & u(k+1) & \cdots & u(k+N-1) \\ u(k+1) & u(k+2) & \cdots & u(k+N) \\ \vdots & \vdots & \ddots & \vdots \\ u(2k-1) & u(2k) & \cdots & u(2k+N-2) \end{pmatrix}$$

Assumptions on Data

- Assumption 1: States visit every dimension.

$$\operatorname{rank} X_p = \operatorname{rank} X_f = n$$

- Assumption 2: Persistently exciting inputs.

$$U_{0|2k-1} = \begin{pmatrix} U_p \\ U_f \end{pmatrix}, \quad \operatorname{rank} \begin{pmatrix} U_p \\ U_f \end{pmatrix} = 2km$$

- Assumption 3: No linear state feedback.

$$\operatorname{span} X_p \cap \operatorname{span} \begin{pmatrix} U_p \\ U_f \end{pmatrix} = \{\phi\},$$

$$\operatorname{span} X_f \cap \operatorname{span} \begin{pmatrix} U_p \\ U_f \end{pmatrix} = \{\phi\}$$

- For a data matrix satisfying Assumptions 1 – 3, any input-output response can be expressed as linear combination of the column vectors of the data matrix:

$$\exists \zeta \in \Re^N \quad \text{such that} \quad \begin{pmatrix} \boldsymbol{u}_f \\ \boldsymbol{u}_p \\ \boldsymbol{y}_p \\ \boldsymbol{y}_f \end{pmatrix} = \begin{pmatrix} U_f \\ U_p \\ Y_p \\ Y_f \end{pmatrix} \zeta$$

- Of particular interest are zero-input responses.

$$\begin{pmatrix} * & 0 & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \\ * & * & * & * \end{pmatrix} = \begin{pmatrix} U_f \\ U_p \\ Y_p \\ Y_f \end{pmatrix} \begin{pmatrix} \zeta_1 & \zeta_2 & \cdots \end{pmatrix}$$

This block should be zero, because all the inputs of both past and future times are zero and the outputs were 0 in the past time.

LQ Decomposition

- Writing the past data matrix as $W_p = \begin{pmatrix} U_p \\ Y_p \end{pmatrix}$,

  we apply LQ Decomposition to the data matrix

$$\begin{array}{ccccc} \begin{pmatrix} U_f \\ W_p \\ Y_f \end{pmatrix} & = & \begin{array}{c} \begin{pmatrix} R_{11} & 0 & 0 \\ R_{21} & R_{22} & 0 \\ R_{31} & R_{32} & 0 \end{pmatrix} \\ \begin{array}{ccc} km & k(m+p) & kp \end{array} \end{array} & \begin{array}{c} \begin{pmatrix} Q_1^T \\ Q_2^T \\ Q_3^T \end{pmatrix} \\ N \end{array} & \begin{array}{l} km \\ k(m+p) \\ kp \\ \; \end{array} \end{array} \qquad (1)$$

where $Q_i^T Q_j = 0, \quad i \neq j$

$Q = \begin{pmatrix} Q_1 & Q_2 & Q_3 \end{pmatrix}$: orthogonal

$R_{11} \in \Re^{km \times km}, R_{22} \in \Re^{k(m+p) \times k(m+p)}$: lower triangular

- From (1), we obtain three equations:

$$U_f = R_{11} Q_1^T \qquad (2)$$

$$W_p = R_{21} Q_1^T + R_{22} Q_2^T \qquad (3)$$

$$Y_f = R_{31} Q_1^T + R_{32} Q_2^T \qquad (4)$$

- Since U_f is of full rank, R_{11} is also full rank and non-singular. Therefore, from (2)

$$Q_1^T = R_{11}^{-1} U_f \qquad (5)$$

- Recall the analysis in the MOESP method,

$$\mathrm{rank} R_{22} = \mathrm{rank} \begin{pmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{pmatrix} = km + n < k(m+p)$$

  Therefore, R_{22} is rank-deficit. Using a pseudoinverse in (3),

$$Q_2^T = R_{22}^{\#} (W_p - R_{21} Q_1^T) \qquad (6)$$

N4SID Method

$$Y_f = R_{31}Q_1^T + R_{32}Q_2^T \qquad (4)$$

$$Q_1^T = R_{11}^{-1}U_f \qquad (5)$$

$$Q_2^T = R_{22}^{\#}(W_p - R_{21}Q_1^T) \qquad (6)$$

- Substituting (5) and (6) into (4),

$$Y_f = R_{31}Q_1^T + R_{32}R_{22}^{\#}(W_p - R_{21}Q_1^T)$$

$$= (R_{31} - R_{32}R_{22}^{\#}R_{21})R_{11}^{-1}U_f + R_{32}R_{22}^{\#}W_p \qquad (7)$$

- Examine the relationship between U_f and W_p.

  Recall $Y_p = O_k X_p + \Psi_k U_p$

  This implies that Y_p is spanned by X_p and U_p. In other words, all the row vectors of Y_p are linear combinations of row vectors involved in X_p and U_p. Therefore,

$$\text{span}W_p = \text{span}\begin{pmatrix} U_p \\ Y_p \end{pmatrix} = \text{span}\begin{pmatrix} U_p \\ X_p \end{pmatrix}$$

- From Assumption 2, $\text{rank}\begin{pmatrix} U_p \\ U_f \end{pmatrix} = 2km$

  This implies no overlap between bases of U_p and U_f.

$$\therefore \text{span}U_p \cap \text{span}U_f = \{\phi\}$$

- From Assumption 3: No linear state feedback,

$$\therefore \text{span}X_p \cap \text{span}U_f = \{\phi\}$$

- Therefore, we conclude that

$$\therefore \text{span}U_f \cap \text{span}W_p = \{\phi\}$$

- Eq.(7) represents Y_f as the sum of two terms that exist in two subspaces having no overlap in their bases.

$$Y_f = \alpha \cdot U_f + \beta \cdot W_p \qquad (8)$$

No overlap in bases: Direct Sum

- In linear algebra, they are called Direct Sum.

N4SID Method

- We can also find a relationship between subspaces spanned by U_f and X_f from:

$$Y_f = O_k X_f + \Psi_k U_f \qquad (9)$$

- From Assumption 3: No linear state feedback

$$\text{span} X_f \cap \text{span} U_f = \{\phi\}$$

Therefore, $Y_f = O_k X_f + \Psi_k U_f$ is a Direct Sum.

- Next, check the relationship between X_f and W_p. One of the elements in X_f can be written as

$$x(k+i) = A^k x(i) + \underbrace{\begin{pmatrix} A^{k-1}B & A^{k-2}B & \cdots & B \end{pmatrix}}_{\bar{\mathcal{C}}_k} \begin{pmatrix} u(i) \\ u(i+1) \\ \vdots \\ u(i+k-1) \end{pmatrix}$$

opposite direction of $\mathcal{C}_k$

- Collectively, this relationship can be expressed as

$$\begin{pmatrix} x(k) & x(k+1) & \cdots & x(k+N-1) \end{pmatrix}$$

$$X_f = A^k X_p + \bar{\mathcal{C}}_k U_p \qquad (10)$$

$$\begin{pmatrix} x(0) & x(1) & \cdots & x(N-1) \end{pmatrix}$$

- Recall $Y_p = O_k X_p + \Psi_k U_p$

$$X_p = O_k^{\#}(Y_p - \Psi_k U_p)$$

Substituting this into (10)

$$X_f = A^k O_k^{\#}(Y_p - \Psi_k U_p) + \bar{\mathcal{C}}_k U_p$$

$$= A^k O_k^{\#} Y_p + (\bar{\mathcal{C}}_k - A^k O_k^{\#} \Psi_k) U_p$$

- X_f is spanned by U_p and Y_p, or

$$X_f \in \text{span} W_p$$

N4SID Method

- So, comparing the two expressions on Y_f, and examining the properties of the subspaces, we can find:

No overlap in bases

(7) $$Y_f = (R_{31} - R_{32}R_{22}^{\#}R_{21})R_{11}^{-1}U_f + R_{32}R_{22}^{\#}W_p$$

Direct Sum

The same subspace

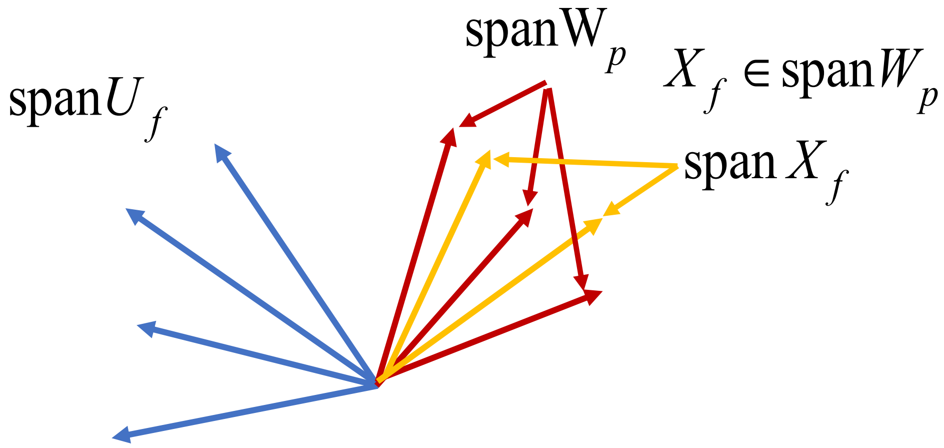
$X_f \in \operatorname{span} W_p$

Direct Sum

(9) $$Y_f = \Psi_k U_f + O_k X_f$$

No overlap in bases

$$\operatorname{span} U_f \cap \operatorname{span} W_p = \{\phi\}$$



Schematic of subspaces

- The conclusion is that the second term in both equations must be the same.

$$\boxed{\therefore O_k X_f = R_{32}R_{22}^{\#}W_p}$$

N4SID Method

- Take Singular-Value Decomposition of the right-hand side

$$\boxed{\therefore O_k X_f = R_{32} R_{22}^{\#} W_p}$$

$$R_{32} R_{22}^{\#} W_p = \begin{pmatrix} U_1 & U_2 \end{pmatrix} \begin{pmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V_1^T \\ V_2^T \end{pmatrix} = U_1 \Sigma_1 V_1^T$$

- This can be split between O_k and X_f

$$X_f = T^{-1} \Sigma_1^{1/2} V_1^T, \quad O_k = U_1 \Sigma_1^{1/2} T$$

- There are two methods for determining system parameters, (A,B,C,D), one is based on the decomposition of X_f and the other on O_k . The following is the former.

- X_f contains a series of states:

$$x(k), x(k+1), \cdots, x(k+N-1)$$

- Based on this series of states and the input-output data, we can form the following 4 matrices.

$$\bar{X}_k = \begin{pmatrix} x(k) & \cdots & x(k+N-2) \end{pmatrix} \in \Re^{n\times(N-1)}$$

$$\bar{X}_{k+1} = \begin{pmatrix} x(k+1) & \cdots & x(k+N-1) \end{pmatrix} \in \Re^{n\times(N-1)}$$

$$\bar{U}_{k|k} = \begin{pmatrix} u(k) & \cdots & u(k+N-2) \end{pmatrix} \in \Re^{m\times(N-1)}$$

$$\bar{Y}_{k|k} = \begin{pmatrix} y(k) & \cdots & y(k+N-2) \end{pmatrix} \in \Re^{p\times(N-1)}$$

- There are related in the state and measurement equations:

$$\begin{pmatrix} \bar{X}_{k+1} \\ \bar{Y}_{k|k} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \bar{X}_k \\ \bar{U}_{k|k} \end{pmatrix}$$

- This has a unique solution,

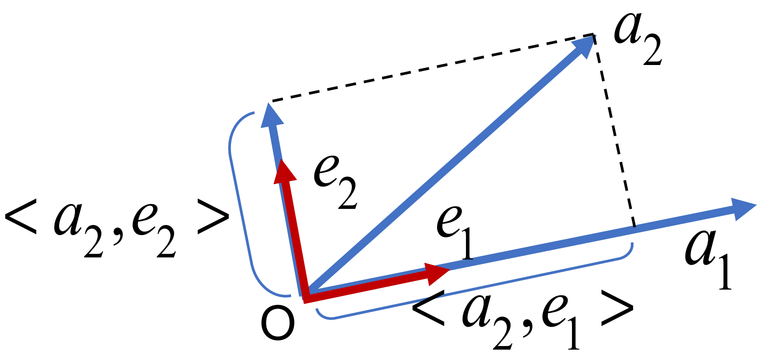
$$\begin{pmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{pmatrix} = \begin{pmatrix} \bar{X}_{k+1} \\ \bar{Y}_{k|k} \end{pmatrix} \begin{pmatrix} \bar{X}_k \\ \bar{U}_{k|k} \end{pmatrix}^T \left(\begin{pmatrix} \bar{X}_k \\ \bar{U}_{k|k} \end{pmatrix} \begin{pmatrix} \bar{X}_k \\ \bar{U}_{k|k} \end{pmatrix}^T \right)^{-1}$$

Comparison between MOESP and N4SID

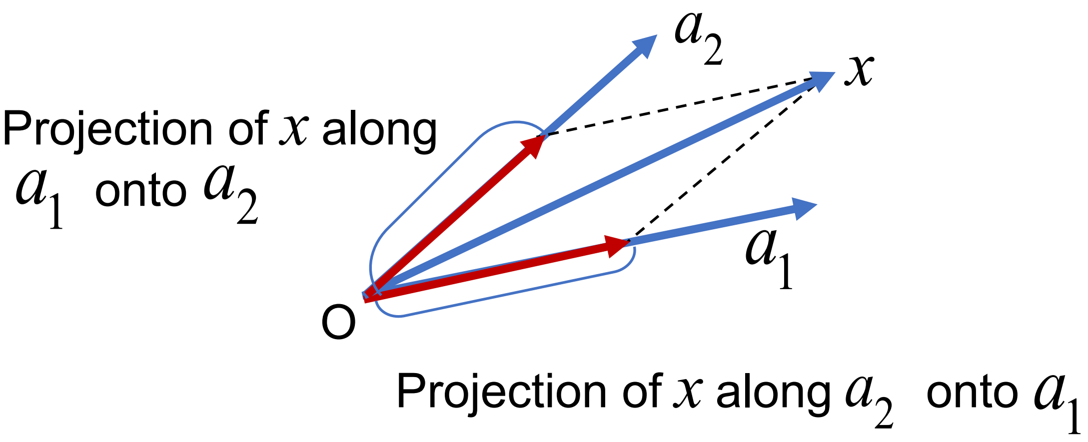
MOESP

N4SID

Mathematical techniques are different.



Orthogonal Projection



Oblique Projection

Almost same performance except for ill-conditioned data