2.160 Identification, Estimation, and Learning

Part 3 Linear System Identification

Lecture 17

Subspace Methods for System Identification: Realization

$$\begin{pmatrix} \bar{x}(t+1) \\ y(t) \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \bar{x}(t) \\ u(t) \end{pmatrix}$$

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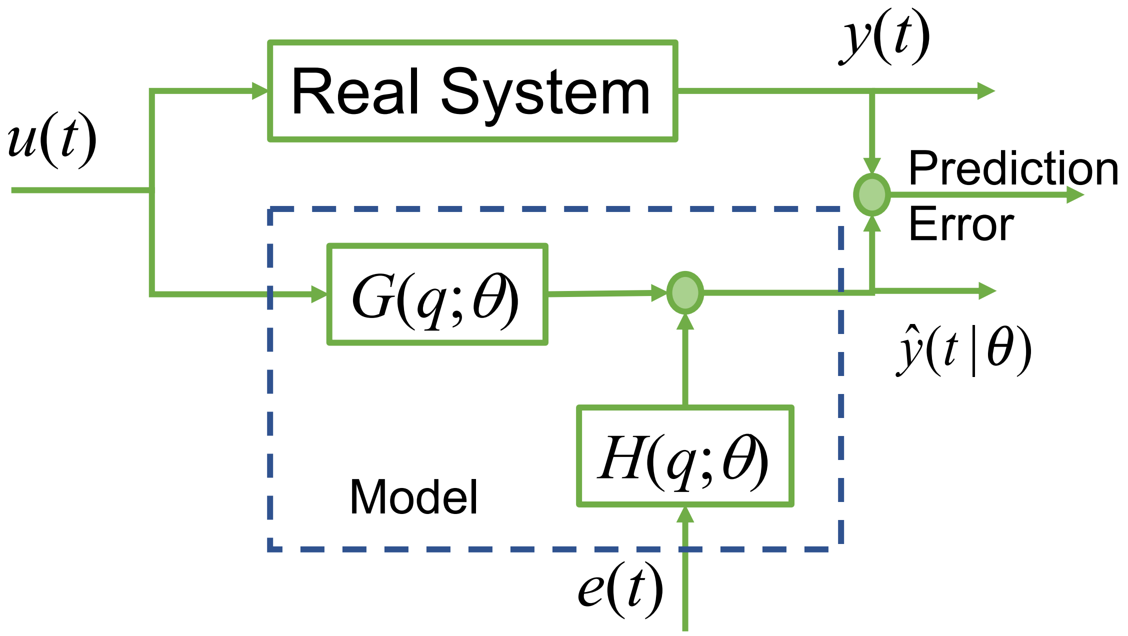
18. Subspace Methods for System identification

Prediction Error Method (PEM)

- Parameters θ are not linearly involved in the predictor $\hat{y}(t|\theta)$ except for ARX and FIR models.
- Local Minima: Non-convex optimization, Repetitive computation; Extended LSE
- Remedy: Instrumental Variables for unbiased (consistent) estimate

$$\hat{\theta}_N = \arg\min_{\theta} \frac{1}{N} \sum (y(t) - \hat{y}(t|\theta))^2$$



Subsystem Methods

- In state space representation of LTI systems, parameter matrices, A, B, C, and D, are linearly involved in state equation and measurement equation. In discrete time,

$$x(t+1) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

- This can be re-arranged to:

$$\underbrace{\begin{pmatrix} x(t+1) \\ y(t) \end{pmatrix}}_{Y(t)} = \underbrace{\begin{pmatrix} A & B \\ C & D \end{pmatrix}}_{\Theta} \underbrace{\begin{pmatrix} x(t) \\ u(t) \end{pmatrix}}_{\varphi(t)}$$

- Replacing all the parameter matrices by Θ, and left hand side by $Y(t)$ and the combination of $x(t)$ and $u(t)$ by $\varphi(t)$, we can find that the above equation can be solved as a standard LSE problem.

Gopinath's Formulation

- B. Gopinath at Bell Laboratories (1969) is the first to formulate the basic algorithm of Subspace methods.
- Consider a multivariate LTI system

$$x(t+1) = Ax(t) + Bu(t) + \eta(t)$$

$$y(t) = Cx(t) + Du(t) + \upsilon(t)$$

where $x \in \Re^{n\times 1}, u \in \Re^{m\times 1}, y \in \Re^{p\times 1}$, $\eta(t)$ and $\nu(t)$ are residues,
and A, B, C, and D, are constant parameter matrices (system matrices) with consistent dimensions.

- Rearranging the above equations,

$$\begin{pmatrix} x(t+1) \\ y(t) \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} + \begin{pmatrix} \eta(t) \\ \upsilon(t) \end{pmatrix} \longleftrightarrow \quad Y = \theta^T \varphi$$

- The parameter matrices are separated from other variables in this expression; a linear regression. The least squares estimate of the parameters is given by

$$\hat{\Theta}_{LS} = \begin{pmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{pmatrix} = \left(\sum_{t=0}^{N-1} \begin{pmatrix} x(t+1) \\ y(t) \end{pmatrix} \begin{pmatrix} x^T(t) & u^T(t) \end{pmatrix} \right) \left(\sum_{t=0}^{N-1} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \begin{pmatrix} x^T(t) & u^T(t) \end{pmatrix} \right)^{-1}$$

$$\begin{pmatrix} x(t+1) \\ y(t) \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} + \begin{pmatrix} \eta(t) \\ \upsilon(t) \end{pmatrix}$$

- The LSE solution is unique, if the regressor covariance is full rank. That is,

$$\operatorname{rank} \begin{pmatrix} x(0) & x(1) & \cdots & x(N-1) \\ u(0) & u(1) & \cdots & u(N-1) \end{pmatrix} = n+m$$

- Residuals η and υ are zero-mean, and their covariance can be obtained by

$$\begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} = \frac{1}{N} \sum_{t=0}^{N-1} \begin{pmatrix} \eta(t) \\ \upsilon(t) \end{pmatrix} \begin{pmatrix} \eta^T(t) & \upsilon^T(t) \end{pmatrix}$$

- If we treat unmodeled dynamics, uncertainties, and some nonlinear effects as noise, the above covariance provides the statistic properties of the noise.
- This is the basic formula of subspace methods found in B. Gopinath’s work in 1969 at the Bell Laboratories.
- The key question, however, is how we can obtain states $x(t)$, $x(t+1)$,… from input and output data. This question is directly related to a System Realization problem.

Realization of LTI Systems

- Let us obtain a state space representation (state and measurement equations) from a given transfer matrix (function). This is called a system realization problem. There are multiple sets of state and measurement equations that produce the same transfer function, i.e. the same input-output behavior.

Straightforward

Realization

$$x(t+1) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

- The particular state space representation we want to realize is the one that uses the lowest dimension of state variables, called Minimal Realization.
- Minimal realization is unique up to a non-singular transformation among the sets of state variables.
- The following are a quick review or a summary of background information required for studying system realization.

Quick Review / Summary of Background Linear System Theory

- The following are a quick review or a summary of background information required for studying system realization.

a. Impulse Response v.s. State Equation

$$x(t+1) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

From the state equation above,

$$\begin{aligned} y(t) &= Cx(t) + Du(t) = C(Ax(t-1) + Bu(t-1)) + Du(t) \\ &= CAx(t-1) + CBu(t-1) + Du(t) \\ &= CA^2x(t-2) + CABu(t-2) + CBu(t-1) + Du(t) \\ \cdots &= CA^t x(0) + Du(t) + \sum_{i=1}^{t} CA^{i-1}Bu(t-i) \end{aligned}$$

Impulse Response: $x(0) = 0$

$u(t) = 0\,; \quad t \neq 0$

The output to the impulse input

$$y(t) = \begin{cases} Du(0): \quad t = 0 \\ CA^{t-1}Bu(0): t > 0 \end{cases} \qquad (1)$$

Let $\{G(0), G(1), G(2), \cdots\}$ be impulse response coefficient matrices

$$y(t) = \sum_{k=0}^{\infty} G(k)u(t-k)$$

For the impulse input, $y(t) = G(t)u(0)$ (2)

Comparing (1) and (2), we find

$$G(t) = \begin{cases} D: \quad t = 0 \\ CA^{t-1}B: t = 1, 2, \cdots \end{cases}$$

b. Observability

- Reconstruct the initial state $x(0)$ from output sequence $y(0), y(1), \cdots, y(k-1)$, assuming no input for all t;

$$u(t) = 0, \forall t$$

$$\begin{aligned} y(0) &= Cx(0) \\ y(1) &= Cx(1) = CAx(0) \\ &\vdots \\ y(k-1) &= CA^{k-1}x(0) \end{aligned}$$

or

$$\begin{pmatrix} y(0) \\ y(1) \\ \vdots \\ y(k-1) \end{pmatrix} = \underbrace{\begin{pmatrix} C \\ CA \\ \vdots \\ CA^{k-1} \end{pmatrix}}_{\mathcal{O}_k \in \Re^{kp \times n}} x(0)$$

If p = 1, k = n, and $\mathcal{O}_n \in \Re^{n \times n}$ is non-singular,

$$x(0) = \mathcal{O}_n^{-1} \begin{pmatrix} y(0) \\ \vdots \\ y(n-1) \end{pmatrix}$$

is uniquely determined.

- The initial state $x(0)$ is determined uniquely, if $\mathcal{O}_k$ is of full column rank.

- In the system identification context, the system order n is often unknown. Therefore, we set k to be strictly larger than n.

- We call matrix $\mathcal{O}_k$ Extended Observability Matrix.

c. Reachability and Controllability

- Consider a discrete-time LTI system of order n. If the initial state $x(0) = 0$ can be transferred to any state at time n by means of a sequence of input, $u(0), u(1), \cdots, u(n-1)$, then the system is called Reachable.

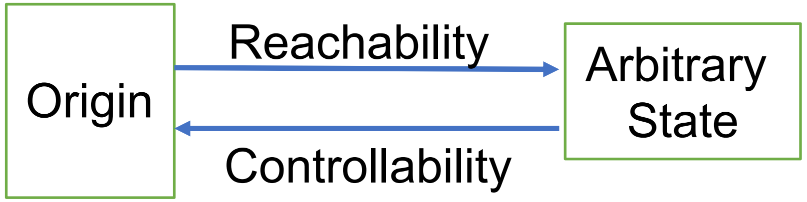
$$x(n) = A^n x(0) + A^{n-1}Bu(0) + A^{n-2}Bu(1) + \cdots + Bu(n-1)$$

$$= A^n x(0) + \underbrace{\begin{pmatrix} B & AB & \cdots & A^{n-1}B \end{pmatrix}}_{\mathcal{C} \in \Re^{n \times nm}} \begin{pmatrix} u(n-1) \\ u(n-2) \\ \vdots \\ u(0) \end{pmatrix}$$

- If matrix $\mathcal{C}$ is of full column rank, then a sequence of input exists that brings the state to an arbitrary state in n steps.

$\mathcal{C} \cdot \begin{pmatrix} u(n-1) \\ u(n-2) \\ \vdots \\ u(0) \end{pmatrix}$ can span the entire n-dimensional space, if $\mathcal{C}$ is of full column rank.

- If matrix A is non-singular, then a reachable system is also controllable.



- In the system identification, we often do not know the system order. We use Extended Reachability Matrix.

$$\mathcal{C}_k \in \Re^{k \times km}, \quad k > n$$

18.2 Ho-Kalman's Method for System Realization

- Ho-Kalman's method is a foundation of Subspace methods, where state and measurement equations are obtained from impulse response coefficient matrices.
- Consider a LTI system that is both observable and reachable, that is, the matrix triple (A,B,C) is minimal.
- We first construct a block Hankel matrix using given impulse response coefficient matrices:

$$H = \begin{pmatrix} G_1 & G_2 & \cdots & G_k \\ G_2 & G_3 & & G_{k+1} \\ \vdots & & \ddots & \vdots \\ G_k & G_{k+1} & \cdots & G_{2k-1} \end{pmatrix} = \begin{pmatrix} CB & CAB & \cdots & CA^{k-1}B \\ CAB & CA^2B & & CA^kB \\ \vdots & & \ddots & \vdots \\ CA^{k-1}B & CA^kB & \cdots & CA^{2k-2}B \end{pmatrix} \in \Re^{kp\times km}$$

Recall

$$G(t) = \begin{cases} D: & t = 0 \\ CA^{t-1}B: & t = 1,2,\cdots \end{cases}$$

where $k > n$

- Interestingly, this Hankel matrix can be decomposed to the extended observability matrix and the extended reachability matrix.

$$H = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{k-1} \end{pmatrix} \begin{pmatrix} B & AB & \cdots & A^{k-1}B \end{pmatrix} = O_k \mathcal{C}_k$$

- Take the Singular Value Decomposition of the Hankel matrix H:

$$H = U\Sigma V^T = \begin{pmatrix} U_1 & U_2 \end{pmatrix} \begin{pmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V_1^T \\ V_2^T \end{pmatrix} = U_1 \Sigma_1 V_1^T$$

where

$$\Sigma_1 = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \qquad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$$

- Note that the rank of the extended observability matrix and that of the reachability matrix are n.
- Obtain the positive square root matrix of Σ_1:

$$\Sigma_1^{1/2} = \begin{pmatrix} \sqrt{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{\lambda_n} \end{pmatrix} \qquad \text{In fact} \quad \Sigma_1^{1/2}\Sigma_1^{1/2} = \Sigma_1$$

- Construct the observability and reachability matrices as follows:

$$\mathcal{O}_k = U_1 \Sigma_1^{1/2} \qquad \text{and} \qquad \mathcal{C}_k = \Sigma_1^{1/2} V_1^T$$

$$\mathcal{O}_k = U_1 \Sigma_1^{1/2} \quad \text{and} \quad \mathcal{C}_k = \Sigma_1^{1/2} V_1^T$$

- The product of these matrices recovers the Hankel matrix H.

$$\mathcal{O}_k \mathcal{C}_k = U_1 \Sigma_1^{1/2} \Sigma_1^{1/2} V_1^T = U_1 \Sigma_1 V_1^T = H$$

- Also, $\mathcal{O}_k = U_1 \Sigma_1^{1/2} T$ and $\mathcal{C}_k = T^{-1} \Sigma_1^{1/2} V_1^T$, too, recovers the Hankel matrix, where T is a non-singular matrix.

- From the above observability and reachability matrices, minimal system matrices (A,B,C) can be determined.

  - From $\mathcal{O}_k = U_1 \Sigma_1^{1/2}$

$$\mathcal{O}_k = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{k-1} \end{pmatrix} \Rightarrow C = \mathcal{O}_k(1:p,:) \in \Re^{p \times n}$$

  - From $\mathcal{C}_k = \Sigma_1^{1/2} V_1^T$

$$\mathcal{C}_k = \begin{pmatrix} B & AB & \cdots & A^{k-1}B \end{pmatrix} \Rightarrow B = \mathcal{C}_k(:,1:m) \in \Re^{n \times m}$$

- To determine the A matrix, we examine the extended observability matrix; Post-multiplying A, we have

$$\begin{pmatrix} C \\ CA \\ \vdots \\ CA^{k-1} \end{pmatrix} A = \begin{pmatrix} CA \\ CA^2 \\ \vdots \\ CA^k \end{pmatrix} = \underbrace{O_{k+1}(p+1:p(k+1),:)}_{O'_{k+1}} \in \Re^{pk\times n}$$

$$\therefore O_k A = O'_{k+1}$$

- Solving this for A: Pre-multiplying O_k^T $(O_k \in \Re^{kp\times n})$

$$\underbrace{O_k^T O_k}_{\text{n x n, Non-singular}} A = O_k^T O'_{k+1} \qquad A = \underbrace{(O_k^T O_k)^{-1} O_k^T}_{\text{Pseudoinverse } O_k^{\#}} O'_{k+1}$$

- Therefore, the minimal system parameters (A,B,C,D) are directly obtained from the Singular Value Decomposition (SVD) of the Hankel matrix H consisting of impulse response coefficients $G(k)$.

18.3 Data Matrices

- ❑ Ho-Kalman's method allows us to determine system parameters in state space, (A, B, C, D), from impulse response coefficients.
- ❑ However, our objective in system identification is to obtain (A, B, C, D) from input-output data.

Input-Output Data $\{(u(t), y(t)) \mid t = 0, 1, 2, \cdots\}$

Impulse Response $G(t), t = 0, 1, 2, \cdots$ $\xrightarrow{\text{Ho-Kalman}}$ State and Measurement Equations (A, B, C, D)

Subspace ID

MOESP, N4SID

- ❑ In Subspace methods, we place data in block Hankel matrix form.

  - ▪ Input Data Matrix

$$U_{0|k-1} = \begin{pmatrix} u(0) & u(1) & \cdots & u(N-1) \\ u(1) & u(2) & & u(N) \\ \vdots & & \ddots & \vdots \\ u(k-1) & u(k) & \cdots & u(k+N-2) \end{pmatrix}$$

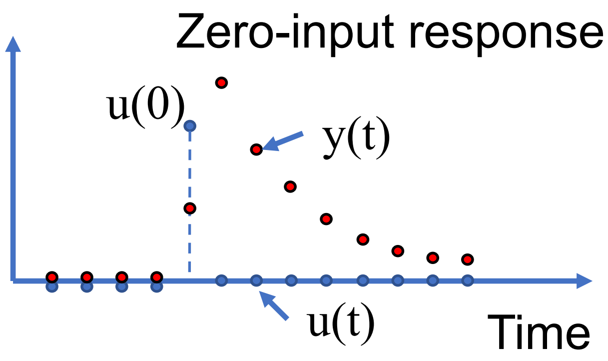
$mk \times N$

  - ▪ Output Data Matrix

$$Y_{0|k-1} = \begin{pmatrix} y(0) & y(1) & \cdots & y(N-1) \\ y(1) & y(2) & & y(N) \\ \vdots & & \ddots & \vdots \\ y(k-1) & y(k) & \cdots & y(k+N-2) \end{pmatrix}$$

$pk \times N$

Example 18-1. Suppose that $y(t) = 0,\ t < 0$. Apply an impulse input at $t = 3$, and observe the response of a LTI system with 3 steps of delay.



$$u = (0,0,0,1,0,0,\cdots), \quad y = (0,0,0,g_0,g_1,g_2,g_3,\cdots)$$

Let $k = 4$, and $N = 8$. We form input and output data matrices, and append them.

$$U_{0|k-1} = \begin{pmatrix} u(0) & u(1) & \cdots & u(N-1) \\ u(1) & u(2) & & u(N) \\ \vdots & & \ddots & \vdots \\ u(k-1) & u(k) & \cdots & u(k+N-2) \end{pmatrix}$$

$$Y_{0|k-1} = \begin{pmatrix} y(0) & y(1) & \cdots & y(N-1) \\ y(1) & y(2) & & y(N) \\ \vdots & & \ddots & \vdots \\ y(k-1) & y(k) & \cdots & y(k+N-2) \end{pmatrix}$$

$$\begin{pmatrix} U_{0|3} \\ Y_{0|3} \end{pmatrix} = \left(\begin{array}{cccc:cccc} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & g_0 & g_1 & g_2 & g_3 & g_4 \\ 0 & 0 & g_0 & g_1 & g_2 & g_3 & g_4 & g_5 \\ 0 & g_0 & g_1 & g_2 & g_3 & g_4 & g_5 & g_6 \\ g_0 & g_1 & g_2 & g_3 & g_4 & g_5 & g_6 & g_7 \end{array}\right)$$

Zero input

This block is the Hankel matrix $H_{4,4}$

- The Hankel matrix is factorized to observability and reachability matrices $H = \mathcal{O}_k \mathcal{C}_k$, from which a minimum system (A,B,C) can be obtained.
- The above input-output data have a special structure: Zero-input response. General input-output data do not have this structure. However, as shown in the following, they can be transformed to the zero-input structure from which system parameter matrices can be determined.

Collective Input-Output Hankel Expression

- From state and measurement equations,

$$y(t) = Cx(t) + Du(t)$$

$$y(t+1) = CAx(t) + CBu(t) + Du(t+1)$$

$$y(t+2) = CA^2x(t) + CABu(t) + CBu(t+1) + Du(t+2)$$

$$\vdots \qquad\qquad \vdots$$

- These equations can be written collectively,

$$\underbrace{\begin{pmatrix} y(t) \\ y(t+1) \\ \vdots \\ y(t+k-1) \end{pmatrix}}_{\boldsymbol{y}_k(t) \quad pk\times 1} = \underbrace{\begin{pmatrix} C \\ CA \\ \vdots \\ CA^{k-1} \end{pmatrix}}_{O_k \quad pk\times n} x(t) + \underbrace{\begin{pmatrix} D & 0 & \cdots & 0 \\ CB & D & & \vdots \\ \vdots & & \ddots & 0 \\ CA^{k-2}B & \cdots & CB & D \end{pmatrix}}_{\Psi_k \quad pk\times mk} \underbrace{\begin{pmatrix} u(t) \\ u(t+1) \\ \vdots \\ u(t+k-1) \end{pmatrix}}_{\boldsymbol{u}_k(t) \quad mk\times 1}$$

- Or, succinctly,

$$\boldsymbol{y}_k(t) = O_k x(t) + \Psi_k \boldsymbol{u}_k(t)$$

This matrix is a block Toeplitz matrix.

Collective Input-Output Hankel Expression

- Note that concatenating $\boldsymbol{y}_k(0) \quad \boldsymbol{y}_k(1) \quad \cdots \quad \boldsymbol{y}_k(N-1)$ yields the block Hankel output matrix,

$$Y_{0|k-1} = \begin{pmatrix} \boldsymbol{y}_k(0) & \boldsymbol{y}_k(1) & \cdots & \boldsymbol{y}_k(N-1) \end{pmatrix}$$

Similarly, $U_{0|k-1} = \begin{pmatrix} \boldsymbol{u}_k(0) & \boldsymbol{u}_k(1) & \cdots & \boldsymbol{u}_k(N-1) \end{pmatrix}$

Also, we define $X_0 \triangleq \begin{pmatrix} x(0) & x(1) & \cdots & x(N-1) \end{pmatrix}$

- The input-output relationship, $\boldsymbol{y}_k(t) = O_k x(t) + \Psi_k \boldsymbol{u}_k(t)$, can be expanded to the block Hankel form,

$$Y_{0|k-1} = O_k X_0 + \Psi_k U_{0|k-1}$$

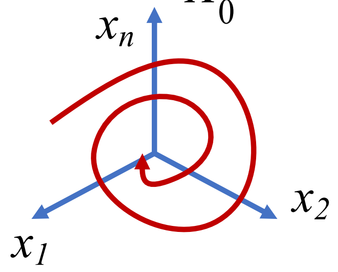
- This is a succinct expression of the following relationship.

$$\underbrace{\begin{pmatrix} y(0) & y(1) & \cdots & y(N-1) \\ y(1) & y(2) & & y(N) \\ \vdots & & \ddots & \vdots \\ y(k-1) & y(k) & \cdots & y(k+N-2) \end{pmatrix}}_{Y_{0|k-1}} = \underbrace{\begin{pmatrix} C \\ CA \\ \vdots \\ CA^{k-1} \end{pmatrix}}_{O_k} \underbrace{\begin{pmatrix} x(0) & x(1) & \cdots & x(N-1) \end{pmatrix}}_{X_0} + \underbrace{\begin{pmatrix} D & 0 & \cdots & 0 \\ CB & D & & \vdots \\ \vdots & & \ddots & 0 \\ CA^{k-2}B & \cdots & CB & D \end{pmatrix}}_{\Psi_k} \underbrace{\begin{pmatrix} u(0) & u(1) & \cdots & u(N-1) \\ u(1) & u(2) & & u(N) \\ \vdots & & \ddots & \vdots \\ u(k-1) & u(k) & \cdots & u(k+N-2) \end{pmatrix}}_{U_{0|k-1}}$$

Assumptions on Data

- ❑ For constructing subspace identification algorithms, we have to make three key assumptions on data.

$$\underbrace{\begin{pmatrix} y(0) & y(1) & \cdots & y(N-1)\\ y(1) & y(2) & & y(N)\\ \vdots & & \ddots & \vdots \\ y(k-1) & y(k) & \cdots & y(k+N-2)\end{pmatrix}}_{Y_{0|k-1}\quad pk\times N} = O_k \underbrace{\begin{pmatrix} x(0) & x(1) & \cdots & x(N-1)\end{pmatrix}}_{X_0} + \Psi_k \underbrace{\begin{pmatrix} u(0) & u(1) & \cdots & u(N-1)\\ u(1) & u(2) & & u(N)\\ \vdots & & \ddots & \vdots \\ u(k-1) & u(k) & \cdots & u(k+N-2)\end{pmatrix}}_{U_{0|k-1}\quad mk\times N}$$



- ❑ Assumption A-1: rank $X_0 = n$.

  The state vector is sufficiently excited, or the system is reachable.

- ❑ Assumption A-2: $\text{rank}\, U_{0|k-1} = mk$

  The input sequence is persistently exciting of order k.

- ❑ Assumption A-3: $\text{rank}\begin{pmatrix} U_{0|k-1} \\ X_0 \end{pmatrix} = mk + n$

Recall $\sum \varphi(t)\varphi^T(t) = (\text{full rank})$

$$\begin{pmatrix} \varphi(0) & \cdots & \varphi(N-1)\end{pmatrix}\begin{pmatrix} \varphi^T(0) \\ \vdots \\ \varphi^T(N-1)\end{pmatrix}$$

$$\Updownarrow$$

$$\begin{pmatrix} u(0) & \cdots & u(N-1)\end{pmatrix}$$

X_0 and $U_{0|k-1}$ are not collinear. No linear state feedback: $u = Kx$.

In other words, the spaces spanned by the input matrix and the state matrix do not intersect.

$$\text{span}\, X_0 \cap \text{span}\, U_{0|k-1} = \{\phi\}$$

Experiments should not be taken with linear state feedback, $u = Kx$.

Transformation to Zero-Input Response

- Under these assumptions, an arbitrary input-output data matrix below can be transformed to a type of the zero-input response form, from which system parameter matrices are determined.

$$W_{0|k-1} = \begin{pmatrix} U_{0|k-1} \\ Y_{0|k-1} \end{pmatrix} \Rightarrow \begin{pmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{pmatrix}$$

Zero-input response

- We need to prove a few Lemmas.

Lemma 18-1.

Suppose that the 3 assumptions are met and rank $O_k = n$ for the following data matrix. Then, the rank of the matrix is:

$$\operatorname{rank}\begin{pmatrix} U_{0|k-1} \\ Y_{0|k-1} \end{pmatrix} = km + n$$

Identity Matrix
rank: km

<u>Proof</u> Rearranging $Y_{0|k-1} = O_k X_0 + \Psi_k U_{0|k-1}$

$$\Rightarrow \underbrace{\begin{pmatrix} U_{0|k-1} \\ Y_{0|k-1} \end{pmatrix}}_{\text{rank: } km+n} = \underbrace{\begin{pmatrix} I_{km} & 0_{km\times n} \\ \Psi_k & O_k \end{pmatrix}}_{\text{rank: } km+n} \begin{pmatrix} U_{0|k-1} \\ X_0 \end{pmatrix}$$

rank: n

rank: $km + n$

- This Lemma implies that, if we delete row vectors in $Y_{0|k-1}$ that are dependent on the row vectors in $U_{0|k-1}$, there remain exactly n independent row vectors in $Y_{0|k-1}$.

Transformation to Zero-Input Response (Continued)

Lemma 18-2

Any input-output pair of length k that satisfies, $\boldsymbol{y}_k(0) = O_k x(0) + \Psi_k \boldsymbol{u}_k(0)$, can be expressed as a linear combination of the column vectors of data matrix $W_{0|k-1}$, which satisfies the 3 assumptions.

That is, there exists a vector $\zeta \in \Re^{N\times 1}$ such that

$$\begin{pmatrix} \boldsymbol{u}_k(0) \\ \boldsymbol{y}_k(0) \end{pmatrix} = \begin{pmatrix} U_{0|k-1} \\ Y_{0|k-1} \end{pmatrix} \zeta, \qquad \text{note: } W_{0|k-1} = \begin{pmatrix} U_{0|k-1} \\ Y_{0|k-1} \end{pmatrix}$$

<u>Proof</u> For an arbitrary input-output pair $\boldsymbol{u}_k(0)$ and $\boldsymbol{y}_k(0)$, there exists an initial state $x(0)$ that satisfies the equation: $\boldsymbol{y}_k(0) = O_k x(0) + \Psi_k \boldsymbol{u}_k(0)$

By assumption, $\operatorname{rank}\begin{pmatrix} U_{0|k-1} \\ X_0 \end{pmatrix} = km + n$, therefore $\begin{pmatrix} \boldsymbol{u}_k(0) \\ x(0) \end{pmatrix} = \begin{pmatrix} U_{0|k-1} \\ X_0 \end{pmatrix} \zeta$

Similar to Lemma 18-1,

$$\begin{pmatrix} \boldsymbol{u}_k(0) \\ \boldsymbol{y}_k(0) \end{pmatrix} = \begin{pmatrix} I_{km} & 0_{km\times n} \\ \Psi_k & O_k \end{pmatrix} \begin{pmatrix} \boldsymbol{u}_k(0) \\ x(0) \end{pmatrix} = \begin{pmatrix} I_{km} & 0_{km\times n} \\ \Psi_k & O_k \end{pmatrix} \begin{pmatrix} U_{0|k-1} \\ X_0 \end{pmatrix} \zeta = \begin{pmatrix} U_{0|k-1} \\ Y_{0|k-1} \end{pmatrix} \zeta$$

Q.E.D.

LQ Decomposition

- From the previous Lemmas, a Zero-Input Response can be created by a linear combination of the column vectors of data matrix $W_{0|k-1}$.

$$\exists \zeta \in \Re^{N\times 1}, \quad s.t. \quad \underbrace{\begin{pmatrix} U_{0|k-1} \\ Y_{0|k-1} \end{pmatrix}}_{W_{0\,|\,k-1}} \zeta = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \tilde{y}(0) \\ \vdots \\ \tilde{y}(k-1) \end{pmatrix},$$

- Repeating this to create sufficient Zero-Input Response vectors, we can form a Hankel matrix from which the system matrices (A,B,C) can be obtained, as demonstrated in Example 18-1.
- However, this transformation of data matrix $W_{0|k-1}$ to the Zero-Input Response form can be achieved by LQ Decomposition.

$$\begin{pmatrix} U_{0|k-1} \\ Y_{0|k-1} \end{pmatrix} \underbrace{\begin{pmatrix} \zeta^1 & \cdots & \zeta^{(m+p)k} \end{pmatrix}}_{Q} = \begin{pmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{pmatrix},$$

- LQ decomposition transforms the data matrix $W_{0|k-1}$ to a Lower Triangular Matrix, which is the form of Zero Input Response

QR Decomposition

- LQ Decomposition is the transpose of so-called "QR Decomposition". An arbitrary rectangular matrix $A \in \Re^{m \times n}$ can be decomposed to an orthonormal matrix Q and an upper triangular matrix in the following form:

$$A = QR = \begin{pmatrix} \underbrace{Q_1}_{n} & \underbrace{Q_2}_{m-n} \end{pmatrix} \begin{pmatrix} R_1 \\ 0 \end{pmatrix} \qquad m \geq n \qquad R_1 = \begin{pmatrix} * & * & \cdots & * \\ 0 & * & * & \vdots \\ \vdots & 0 & * & * \\ 0 & \cdots & 0 & * \end{pmatrix} \in \Re^{n \times n}$$

- Matrix Q consists of unit-length column vectors that are orthogonal to each other.

$$Q^T Q = \begin{pmatrix} Q_1^T \\ Q_2^T \end{pmatrix} \begin{pmatrix} Q_1 & Q_2 \end{pmatrix} = \begin{pmatrix} Q_1^T Q_1 & Q_1^T Q_2 \\ Q_2^T Q_1 & Q_2^T Q_2 \end{pmatrix} \begin{pmatrix} I_n & 0 \\ 0 & I_{m-n} \end{pmatrix} = I_m$$

- MATLAB code: qr(A), (Q, R) = qr(A) returns an orthonormal matrix Q and an upper triangular matrix of the above form.
- There are effective algorithms to obtain the QR factorization of a rectangular matrix.
  - Gram-Schmidt procedure – numerically not stable
  - Householder Reflection – widely used method

LQ Decomposition of Data Matrix $W_{0|k-1}$

- We apply the transpose of the QR decomposition form to the data matrix $W_{0|k-1}$.

$$A = QR \rightarrow A^T = R^T Q^T \qquad \begin{pmatrix} U_{0|k-1} \\ Y_{0|k-1} \end{pmatrix} = \begin{pmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} Q_1^T \\ Q_2^T \end{pmatrix},$$

- The two well-known Subspace Algorithms can be derived from this LQ Decomposition.

  - MOESP (<u>M</u>ultivariable <u>O</u>utput <u>E</u>rror <u>S</u>tate s<u>P</u>ace)
  - N4SID (<u>N</u>umerical algorithm for <u>S</u>ubspace <u>S</u>tate <u>S</u>pace <u>S</u>ystem <u>I</u>dentification) --- read “Enforce It!”